CS 6170 Computational Topology: Topological Data Analysis **University of Utah**
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19.1 Review

Let M be a d-dimensional manifold and $f : M \to \mathbb{R}$ be a real valued function on M. Recall

1. Critical points of f have partial derivatives equal to 0
2. Degenerate critical points have second partial derivatives equal to 0, and nondegenerate critical points have a Hessian matrix with nonzero determinant

Lemma 19.1 (Morse Lemma)**.** *If u is non-degenerate critical point then there exists local coordinates with $u = (0, \cdots, 0)$ and*

$$f(x) = f(u) - x_1^2 - \cdots - x_q^2 + x_{q+1}^2 + \cdots + x_n^2 \tag{19.1}$$

f is a Morse function if

1. All critical points are non-degenerate
2. f has distinct critical values

The index of a critical point x, denoted index(x), is the number of negative coefficients in the quadratic local coordinates around x.

Lemma 19.2 (Morse Inequalities)**.** *1. Weak version*

$$c_q > \beta_q \tag{19.2}$$

2. Strong version

$$\sum_{q=0}^{j}(-1)^{j-q}c_q \geq \sum_{q=0}^{j}(-1)^{j-q}\beta_q \tag{19.3}$$

where c_q is the number of critical points with index q, and β_q is the q^{th} Betti number.

19.2 Piecewise Linear functions

Let K be a simplicial complex with n vertices $(u_1, \cdots, u_n)$ and real (distinct) values at each vertex

$$f : |K| \to \mathbb{R} \tag{19.4}$$

Define a real valued function on K as

$$f(x) = \sum_i b_i(x) f(u_i) \tag{19.5}$$

where $b_i(x)$ are the barycentric coordinates. Suppose we have an ordering $u_1, \cdots, u_n$ such that

$$f(u_1) < \cdots < f(u_n) \tag{19.6}$$

Let K_i be the subcomplex of first i vertices

$$\emptyset = K_0 \subset \cdots \subset K_n = K \tag{19.7}$$

Definition 19.3. *1. The star of u: the set of co-faces of u, denoted St u*

2. The closed star of u: the closure of St u, denoted $\overline{\text{St}}$ u,

3. The lower star of u: St$_-$ $u = \{\sigma \in \text{St } u | x \in \sigma \Rightarrow f(x) \subseteq f(u)\}$

4. Link of vertex: the set of simplices in the closed star but not in the star, denoted Lk u

5. Lk$_-$ $u = \{\sigma \in \text{Lk } u | x \in \sigma \Rightarrow f(x) \leq f(u)\}$

The filtration given from the piecewise linear function on K is called the lower star filtration.

We want to classify vertices based on the reduced Betti number of the lower link Lk$_-$.

1. Regular vertex: $\tilde{\beta}_0 = \tilde{\beta}_1 = 0$
2. Minima: $\tilde{\beta}_{-1} = 1$ (ie the lower link is empty)
3. Maxima: Lk$_-$ u =Lk u, $\tilde{\beta}_1 = 1$

We say a PL critical vertex is simple of index q if the lower link has reduced homology of $(q-1)$-sphere ($\tilde{\beta}_{q-1} = 1$ is the only nonzero Betti number).

19.3 PL Morse function

Definition 19.4. *A function $f : |K| \to \mathbb{R}$ is a PL Morse function if*

1. *each vertex is PL regular or PL simple critical*
2. *function values at vertices are distinct*

Lemma 19.5 (PL Morse Inequalities). *1. Weak:*

$$c_q \geq \beta_q(K) \tag{19.8}$$

2. Strong:

$$\sum_{q=0}^{j} (-1)^{j-q} c_q \geq \sum_{q=0}^{j} (-1)^{j-q} \beta_q \tag{19.9}$$

where c_q is the number of vertices of index q, and β_q is the q^{th} Betti number.

19.4 Computing Reeb graph

Let $f : |K| \to \mathbb{R}$, where K is a triangulation of a 2-manifold. Sort the vertices of K so that $f(u_i) < f(u_{i+1})$. Suppose that $f(u_i) < s < f(u_{i+1})$. Then $f^{-1}(s)$ is some 1-manifold.

1. If u is a minimum $\to$ add degree 1 node to graph.
   $\to$ create new arc as a list of triangles in lower star of u
2. If u is regular vertex, triangles in its lower star form a sequence in existing arc list. Replace lower star triangles with upper star triangles
3. If u is a saddle, triangles in lower star of u form two sequences in existing arc list $\Rightarrow$ either split a list $\to$ in 2 or merge two lists into 1