Lecture 18: March 9, 2017

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18.1 Morse Function

In short: A non-degenerate height function

18.1.1 Review

Given $f : \mathbb{M} \to \mathbb{R}$, where $\mathbb{M}$ is a smooth $d-$dimensional manifold, and a local coordinate system[1] $(x_1, x_2, \dots, x_d)$ in a neighborhood of x then x is a critical point if and only if all of its first-order partial derivatives go to zero, $\frac{\partial f(x)}{\partial x_1} = \frac{\partial f(x)}{\partial x_2} = \cdots = \frac{\partial f(x)}{\partial x_d} = 0$.

To identify the different types of critcal points we look at the second derivatives. Since we are in a $d-$dimensional space, we can represent all the second derivatives in a matrix called the Hessian:

$$H(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_d} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_d \partial x_1} & \frac{\partial^2 f(x)}{\partial x_d \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_d \partial x_d} \end{bmatrix}$$

If the determinant of this matrix, $H(x)$ is non-zero then the point x is non-degenerate. This will always occur if the matric $H(x)$ is non-singular.

Example 1: If $\det(H(x)) = 0$ then x is degenarate. It is stated in the book that

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x_1, x_2) \to x_1^3 - 3x_1x_2^2$$

is degenerate when $(x_1, x_2) = (0, 0)$. Computing the first partials: $\frac{\partial f(x)}{\partial x_1} = 3x_1^2 - 3x_2^2$ and $\frac{\partial f(x)}{\partial x_2} = -6x_1x_2$. Then the Hessian is:

$$H(x) = \begin{bmatrix} 6x_1 & -6x_2 \\ -6x_2 & -6x_1 \end{bmatrix}$$

The determinant of $H(x)$ is given by $\det(H(x)) = -36x_1^2 - 36x_2^2$. Thus the $\det(H(0,0)) = 0$. Therefore $(x_1, x_2) = (0, 0)$ is indeed degenerate and is often referred to as a monkey saddle.

[1]Per wikipedia: "A simple example (of local coorsinates) is using house numbers to locate a house on a street."

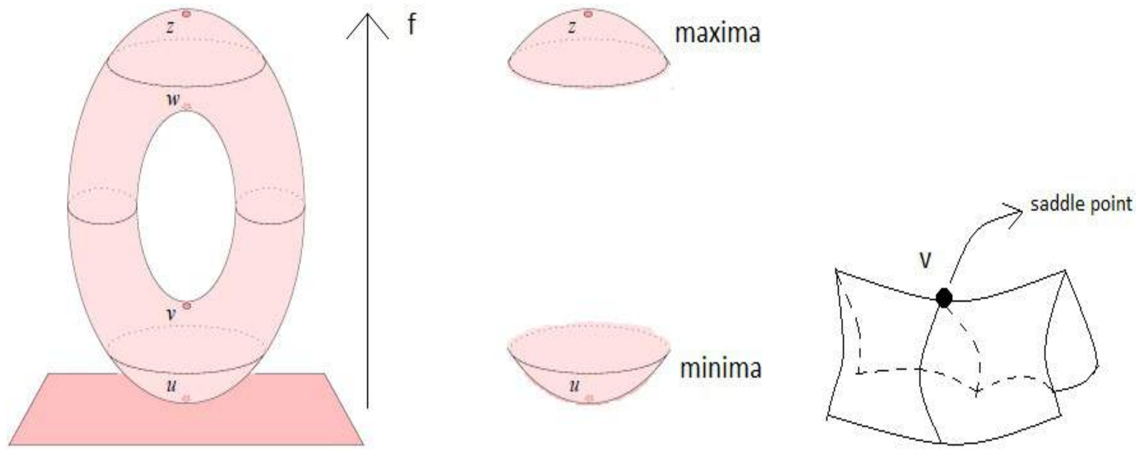

Figure 18.1:

Example 2: Another example was given in class of the torus lying down. A height function defined on this is degenerate, there number of negative eigenvalues do not necessairly determine the local behavior. Put another way, we cannot determine the nature of the critical "point" by approaching it along a straight line.[2] Since the lying down torus is degenerate there does not exist a morse function.

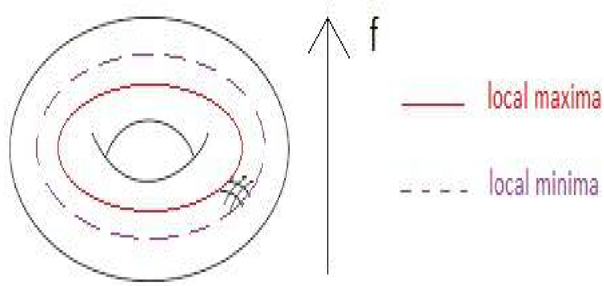

Figure 18.2:

Thinking in terms of eignevalues is useful, because our goal with Morse Functions is to be able to describe global structure using local features. We can do this only if the critical points are non-degenerate because then its nature is predicatable for us.

18.1.2 Morse Lemma

Lemma 18.1. *Let u be a non-degenerate critical point of $f : \mathbb{M} \to \mathbb{R}$. There are local coordinates with $u = (0, 0, \dots, 0)$ such that:*

$$f(x) = f(u) - x_1^2 - \cdots - x_q^2 + x_{q+1}^2 + \cdots + x_d^2$$

for every point $x = (x_1, x_2, \dots, x_d)$ in a small neighborhood of u.

[2] For more intuition into this see: `https://www.math.upenn.edu/~kazdan/260S12/notes/several-var-intuition.pdf`

Lemma 18.1 describes the behavior of a function near the critical points. It also offers a corollary that non-degenerate critical points are isolated. This gives further justification for why a lying down torus would potentially have degenerate critical "points", because the critical points are defined by circles, thus they cannot be isolated. One must be careful with this logic though.

Index

Definition: The number of negative signs in the quadratic polynomial is called the *index* of the critical point.

Example: Given the function f and critical point u in lemma 18.1, $index(u) = q$.

The index classifies $d + 1$ types of non-degenerate critical points. It is possible to have 0 negative signs, 1 negative sign, so on up to d negative signs. Thus:

$$\underbrace{\underbrace{\underbrace{\text{0 negatives}}_{\text{1 critical point}}}_{\text{1 critical point}}\ \underbrace{\underbrace{\text{1 negative}}_{\text{1 critical point}}\ \underbrace{\text{2 negatives}}_{\text{1 critical point}}\ \underbrace{\cdots}_{\cdots}\ \underbrace{d\text{ negatives}}_{\text{1 critical point}}}_{d\text{ critical points}}}_{d+1\text{ critical points}}$$

Let's look at the indexes on the standing torus:

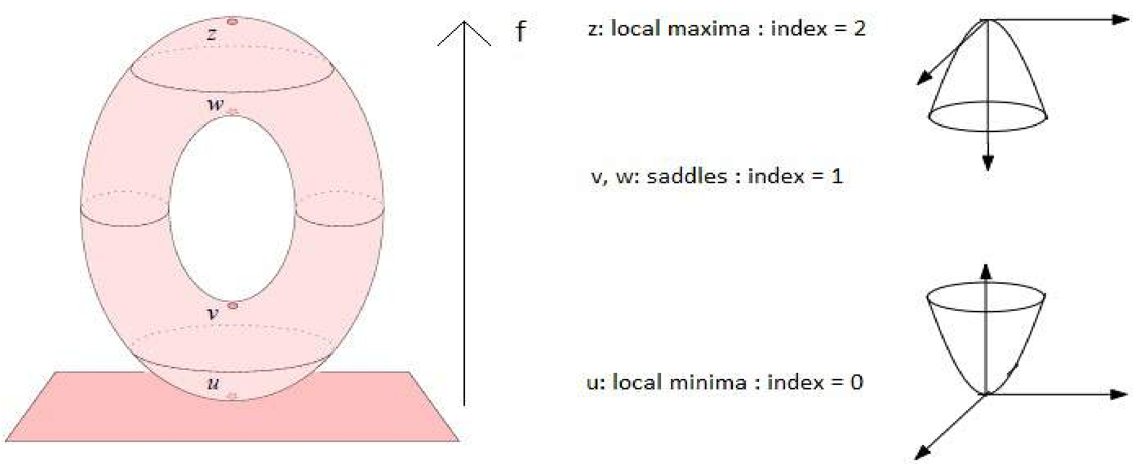

Figure 18.3:

Example: Since the standing torus is a $2-$manifold. $d = 2 \Rightarrow d+1 = 3$, thus there are three types of critical points.

1. A minimum with an index$= 0$ therefore locally $f(x) = f(t) + x_1^2 + x_2^2$
2. Two saddle points each with an index$= 1$, locally:
   (a) for v, $f(x) = f(v) + x_1^2 - x_2^2$
   (b) for w, $f(x) = f(w) - x_1^2 + x_2^2$
3. A maximum with an index$= 2$, locally $f(x) = f(z) - x_1^2 - x_2^2$

18.1.3 Handle attachment

The dimension of the handle attachment is the same as the index we pass. We utilize this relationship in the process of attaching a handle.

18.1.3.1 Attaching a handle

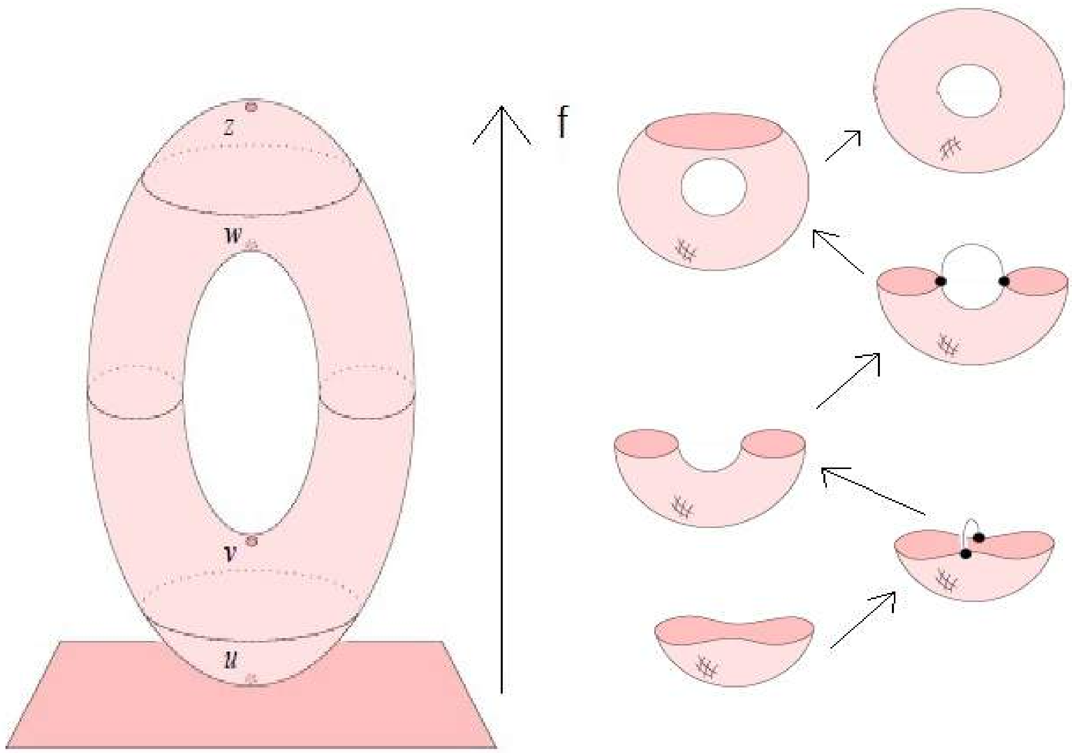

Figure 18.4:

Consider v a critical point with index q, and manifolds $\mathbb{M}_b \simeq \mathbb{M}_a$ with a $q-$ dimensional handle attached and $a < v < b$.

By attach a handle we mean:

B^q : is a $q-$dimensional unit ball
S^{q-1} : is the boundary of B^q

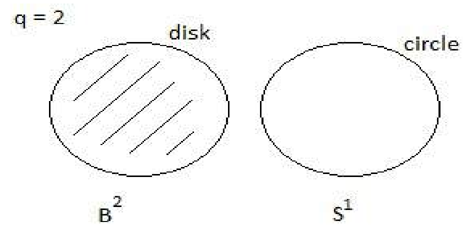

Figure 18.5:

$g : s^{q-1} \to$ boundary $\mathbb{M}_q$, where g is continuous

To attach a handle to $\mathbb{M}_a$, we identify each point $x \in S^{q-1}$ with its image under $g \in$ boundary $\mathbb{M}_a$. In other words we map the boundary of the ball (the handle we are attaching) to the boundary of the surface to which we are attaching a handle.

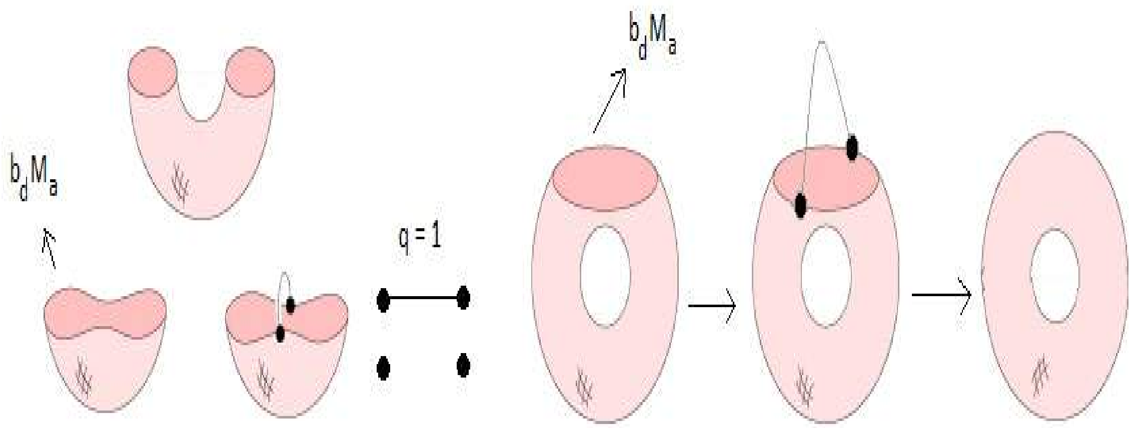

Figure 18.6:

Claim: A consequence of the Morse Lemma is that all non-degenrate critical points are isolated.

18.1.4 Morse Function

Definition: A *Morse function* is a smooth, real-valued function on a manifold, $f : \mathbb{M} \to \mathbb{R}$ such that:

1. All critical points are non-degenerate
2. The critical points have distinct function values[3]

Note: All "height" functions on a sphere are Morse functions.

[3] Distinctness is occationally dropped, because in data analysis we deal with finitie sets so it is possible to perturb the points slightly to acheive distinctness.

Morse Inequality

The Morse inequalities compare the number of critical points to the Betti number.

Given:
C_q : denote the number of non-degnerate critical points with index q.
β_q : is the q^{th} Betti number.
$\mathbb{M}$ is a $d-$dimensional manifold.
$f : \mathbb{M} \to \mathbb{R}$ is a Morse function.

18.1.4.1 Weak Morse Inequality

$$C_q \geq \beta_q(\mathbb{M}\forall q$$

Example: For the torus:

Critical Points	$C_0 = 1$	$C_1 = 2$	$C_2 = 1$
Betti Number	$\beta_0 = 1$	$\beta_1 = 2$	$\beta_2 = 1$

18.1.4.2 Strong Morse Inequality

$$\Sigma_{q=0}^{j}(-1)^{j-q}C_q \geq \Sigma_{q=0}^{j}(-1)^{j-q}\beta(\mathbb{M})$$

The equality holds when $j = d$.

What is so special about these inequalities is that they relate the local (number of critical points) with the global (Betti number).

Example: For the torus:

$$C_0 - C_1 + C_2 = 0 = \beta_0 - \beta_1 + \beta_2 = \chi(\mathbb{M})$$

where the alternating sum of the Betti number, $\chi(\mathbb{M})$ is called the Euler characteristic of the manifold.

18.1.5 Reeb Graph

A Reeb graph is the skeleton representation of the underlying space with respect to a function f. They distill the shape of the data.

Reeb Graph of a Morse function

The Reeb graph of a Morse function is considered a summary of the Morse function on smooth manifolds. The idea boils down to looking at all the level sets and shrink the connected component of the level set to a point.

Some quick facts about Reeb graphs:

- Reeb Graphs are denoted: $R(\mathbb{X}, f)$, where $\mathbb{X}$ is the underlying space and f is a function on $\mathbb{X}$.
- The connected component of a level set is sometimes called a contour.
- Reeb graphs are generic versions of contour tree (A contour tree is the spefic case of a Reeb graph when there are no loops present).
- The degree of local minima/maxima in a Reeb graph is 1.
- The degree of saddle points is 3.
- All other points have degree 2.

It is possible to query for the contour by looking at how many edges are at the lvel this will then tell us the number of connected components.

Reeb graphs are often utilized to summarise the underlying shape in shape classification applications.

Example 1: For the torus

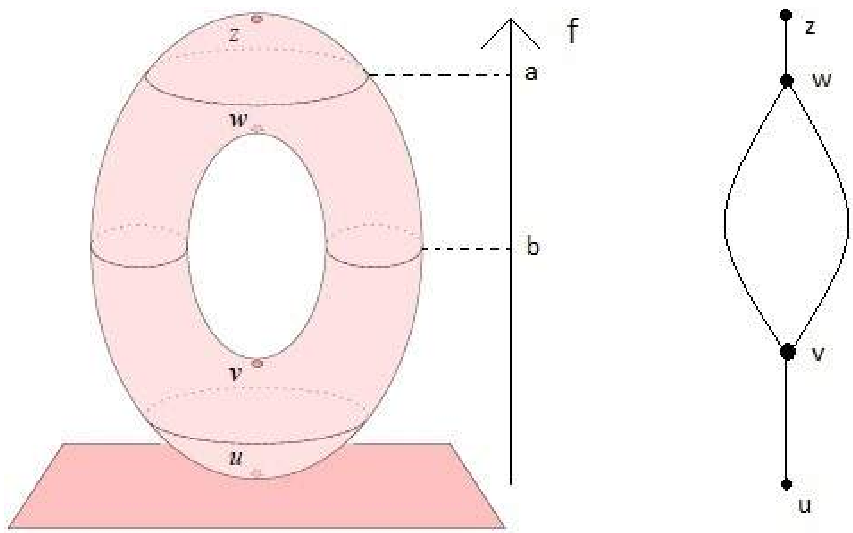

Figure 18.7:

Example 2: Double torus:

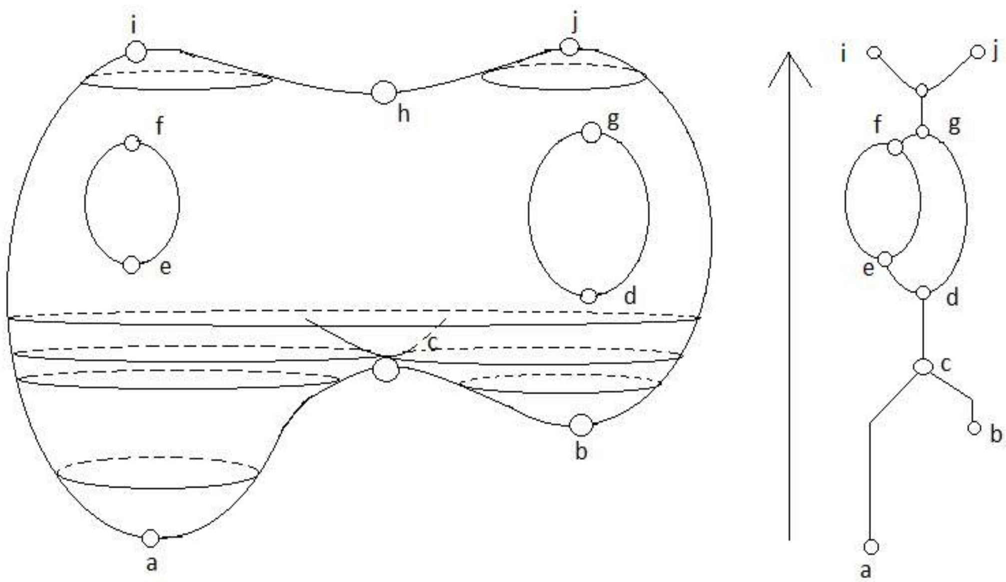

Figure 18.8: