CS 6170 Computational Topology: Topological Data Analysis	**University of Utah**
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14.1 Homology vs. Coholomolgy

14.1.1 Motivation

We can use dimensionality reduction techniques to project higher order point clouds into lower dimensional spaces. This allows us to compute homology in a simpler space. However, this can cause issues including the loss of information and distortion of the projected points. For example, the number of intersections can change based on the angle used to project an $\mathbb{R}^3$ point cloud to a two-dimensional plane. An example of this is shown in Figure 14.1.

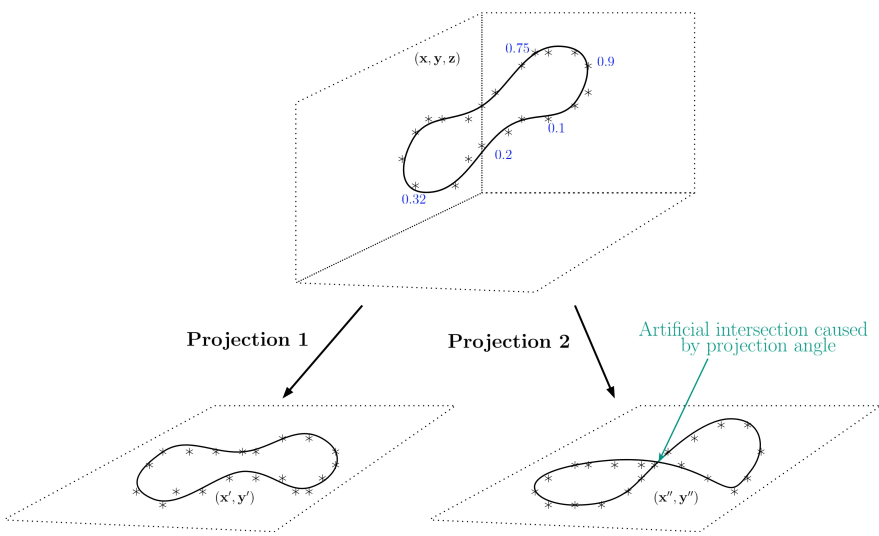


Figure 14.1: Projection examples from $\mathbb{R}^3$ to $\mathbb{R}^2$

In simplicial homology, we can define a chain $c \in C_p$ such as $c = 12 + 23 + 34$. Using simplicial cohomology, we can define a related cochain $c^* \in C^p$ that defines a parameterization of c as $c^* : c \to \mathbb{Z}_2$ where the parameterization maps an input chain c to a coefficient. This allows the use of a function f to map the point cloud $\mathbb{X}$ to a range such as $S^1 = [0, 1]$. Using this mapping, we can generate values as we move along the underlying points. By assigning increasing values (blue in Figure 14.1) to the points, we can determine that there is in fact no intersections in the original point cloud. Using this approach, we can also use multiple functions to determine the amount of tunnels in a complex shape such as the 'bouquet of flowers' in Figure 14.2.

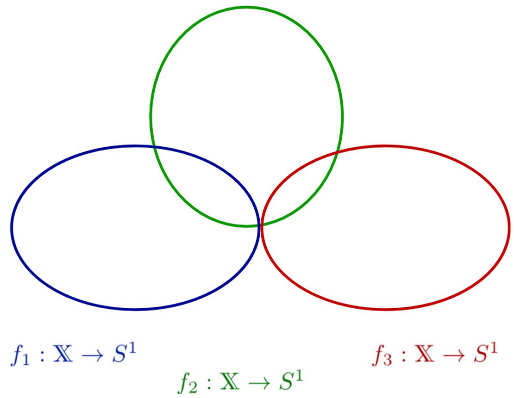


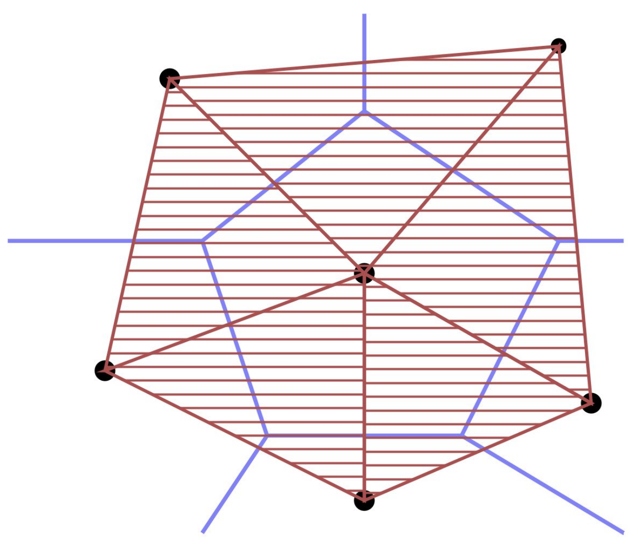
Figure 14.2: Mapping three rings using multiple parameterizations

14.1.2 Duality

We can say that there exists a duality between homology and cohomology. The notion of homology gives a “geometric“ representation to a set of simplicial complexes, where cohomology provides an “algebraic“ interpretation“. This relation is shown in Figure 14.3. Another example of a duality would be the Voronoi diagram (blue) and Delaunay complex (red) which is shown in Figure 14.4.

$$\begin{array}{ccc} \textbf{Homology} & \longleftrightarrow & \textbf{Cohomology} \\ \text{“}\textit{Geometric}\text{”} & & \text{“}\textit{Algebraic}\text{”} \end{array}$$

Figure 14.3: Homology and Cohomology Duality



Voronoi Diagram $\longleftrightarrow$ Delaunay Complex

Figure 14.4: Another example of a Duality

14.2 Cohomology Examples

Let κ be the simplicial complex in Figure 14.5:

$$\kappa = \begin{cases} v_0, v_1, v_2, v_3 \\ e_1, e_2, e_3, e_4, e_5 \\ \Delta_1, \Delta_2 \end{cases}$$

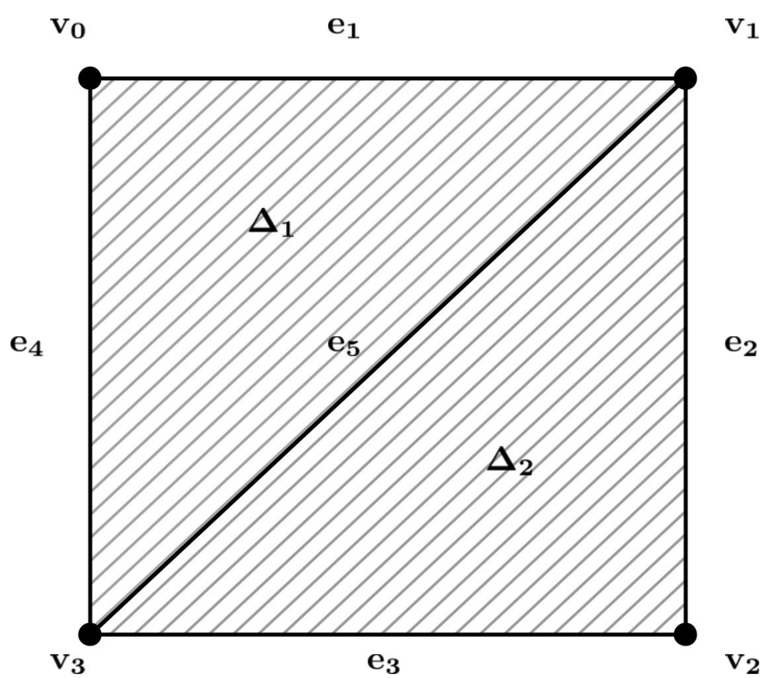


Figure 14.5: Simplicial Complex κ

14.2.1 Chains

As we have seen before, we can take examples of p-chains from this simplicial complex:

0-chains (b): $b_1 = v_1,\ b_2 = v_1 + v_2,\ b_3 = v_1 + v_2 + v_3$.
1-chains (a): $a_1 = e_1,\ a_2 = e_2,\ a_3 = e_1 + e_2 + e4 + e_4 + e_5$
2-chains (c): $c_1 = \Delta_1,\ c_2 = \Delta_1 + \Delta_2$

Definition 14.1. *A k-chain with a single element is known as an* ***elementary k-dimensional chain*** *or elementary k-chain.*

For example, b_1 is an elementary 0-chain and a_1 is an elementary 1-chain.

We can also take the boundary of such chains:

$$\partial(c_2) = \partial(\Delta_1) + \partial(\Delta_2) = (e_1 + e_4 + \cancel{e_5}) + (e_2 + e_3 + \cancel{e_5}) = e_1 + e_2 + e_3 + e_4 \tag{14.1}$$

14.2.2 Cochains

We also have ***k*-cochains** and **elementary *k*-cochains** which are the duals of *k*-chains and elementary *k*-chains:

0-cochains (β):
$v_0^*,\ v_1^*,\ v_2^*,\ v_3^*$ (elementary 0-cochains)
$\beta_0 = v_0^* + v_1^*$

These cochains give mappings to their equivalent chains:

$$\begin{cases} v_0^*(v_0) = 1 \\ v_0^*(v_n) = 0, \text{ where } n \neq 0 \end{cases}$$

Similarly for the other dimensions:

1-cochains (α):
$e_1^*,\ e_2^*,\ e_3^*,\ e_4^*,\ e_5^*$ (elementary 1-cochains)
$\alpha_0 = e_1^* + e_2^*,\ \alpha_1 = e_1^* + e_5^*$

The mappings are similar to their 0-dimensional counterparts: $e_1^*(e_1) = 1, e_1^*(e_2) = 0, \cdots$

2-cochains (γ):
$\Delta_1^*,\ \Delta_2^*$ (elementary 2-cochains)

$\gamma_0 = \Delta_1^* + \Delta_2^*$

And the corresponding mappings: $\Delta_1^*(\Delta_1) = 1,\ \Delta_1^*(\Delta_2) = 0,\ \Delta_2^*(\Delta_1) = 0,\ \Delta_2^*(\Delta_2) = 1$

Cochains also hold under the distributive property of addition:

$\beta_0(v_0) = (v_0^* + v_1^*)(v_0) = v_0^*(v_0) + v_1^*(v_0) = 1$

14.2.3 Coboundary

A duality called the **coboundary** exists as a parallel to the boundary.

Let the coboundary operator be symbolized as δ. This gives us the following definition:

If $c = \sum g_i \sigma_i^*,\ g_i \in \mathbb{Z}_2 = \{0, 1\}$ where σ_i^* can be v_i^*, e_i^*, or Δ_i^*, and c is a linear combination of coefficients over the elementary cochain σ_i^*.

Then $\delta\ c = \sum g_i(\delta\ \sigma_1^*)$

We can take the coboundary of the *p*-simplex σ^*:

$\delta\ \sigma^* = \sum \varepsilon_j \tau_j^*$

If σ^* corresponds to a p-simplex σ, then τ_j is a (p + 1)-simplex having σ as a face.

Examples:

$\delta\ e_5^* = \Delta_1^* + \Delta_2^*$

$\delta\ v_1^* = e_1^* + e_2^* + e_5^*$

Similar to boundaries, the coboundary of a coboundary is zero:

$\delta(\delta(v_1^*)) = \delta(e_1^*) + \delta(e_2^*) + \delta(e_5^*) = \cancel{\Delta_1^*} + \cancel{\Delta_2^*} + (\cancel{\Delta_1^*} + \cancel{\Delta_2^*}) = 0$

14.2.4 Cohomology Groups

In Homology we have the p-cycle group Z_p and p-boundary group B_p. In Cohomology we have p-cocycle group Z^p and p-coboundary group B^p. A cocycle is a p-chain that has a coboundary of zero. A p-dimensional cochain is a coboundary if there exists a (p+1)-chain that is coboundary of.

Examples:

1. Δ_1^* is a cocycle because $\delta\ \Delta_1^* = 0$
2. Δ_1^*, Δ_2^* are coboundaries, since $\delta\ e_1^* = \Delta_1^*,\ \delta\ e_3^* = \Delta_2^*$
3. $a^* = e_1^* + e_5^* + e_3^*$ is a cocycle and a coboundary:
   $\delta(a^*) = \delta(e_1^*) + \delta(e_5^*) + \delta(e_3^*) = \Delta_1^* + (\Delta_1^* + \Delta_2^*) + \Delta_2^* = 0$ and $\delta(v_1^*) = a^*$
4. A zero dimensional cochain $\beta = v_0^* + v_2^* + v_3^* + v_1^*$ is a cocycle:

$$\begin{aligned}\delta(\beta) &= \delta(v_0^*) + \delta(v_2^*) + \delta(v_3^*) + \delta(v_1^*)\\ &= (e_1^* + e_4^*) + (e_3^* + e_2^*) + (e_3^* + e_4^* + e_5^*) + (e_1^* + e_2^* + e_5^*)\\ &= 0\end{aligned}$$

   Therefore β is a cocycle but not a coboundary.

We can define a cohomology group as $H^p = Z^p/B^p$, or the p-cocycle group *mod* the p-coboundary group.