CS 6170 Computational Topology: Topological Data Analysis **University of Utah**
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13.1 Review: Computing Homology

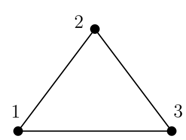
Let ∂_p be the p-th boundary matrix. After reduction (row and column operation), ∂_p turns to Smith's Normal Form (SNF). Let $\mathcal{N}_p$ denote this matrix. $\mathcal{N}_p$ has the following form:

$$\text{rank } B_{p-1} \left\{ \underbrace{\begin{bmatrix} 1 & \cdots & 0 & \overbrace{\cdots}^{\text{rank } Z_p} \\ & \ddots & & \\ 0 & & 1 & \\ \vdots & & & \\ & \cdots & & \cdots \end{bmatrix}}_{\text{rank } C_p} \right\} \text{rank } C_{p-1}$$

Regarding Betti numbers, we have: $\beta_p = \text{rank } \mathcal{Z}_p$ - rank $\mathcal{B}_p$

13.1.1 Example

Consider the following simplicial complex which consists of three edges and three vertices (no triangle).



For this simplicial complex, we have:

- **Reduced Homology:** $\widetilde{\beta_0} = \beta_0 - 1 = 1$, $\widetilde{\beta_1} = \beta_1 = 1$
- **Homology:** $\beta_0 = 1$, $\beta_1 = 1$ (number of tunnels)

Now, we prove these. The dimension-zero boundary matrix ∂_0 and its corresponding SNF matrix $\mathcal{N}_0$ are:

$$\partial_0 = \begin{array}{c} \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\Rightarrow \begin{matrix} & 1 & 2(+1) & 3(+1) \\ 1 & \boxed{1} & \not{1} & \not{1} \end{matrix}$$

$$\mathcal{N}_0 = \begin{matrix} & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \end{matrix}$$

The dimension-one boundary matrix ∂_1 and its corresponding SNF matrix $\mathcal{N}_1$ are:

$$\partial_1 = \begin{matrix} & 12 & 23 & 31 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 1 & 1 \end{matrix}$$

$$\Rightarrow \begin{matrix} & 12 & 23 & 31(+23) \\ 1 & \boxed{1} & 0 & 1 \\ 2(+1) & \not{1} & \boxed{1} & \not{1} \\ 3(+2) & 0 & \not{1} & \not{1} \end{matrix}$$

$$\mathcal{N}_1 = \begin{matrix} & 12 & 23 & 31 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{matrix}$$

- $\widetilde{\beta_0}$ =rank$\mathcal{Z}_0-$rank$\mathcal{B}_0 = 0$
- $\widetilde{\beta_1} = \beta_1$ =rank$\mathcal{Z}_1-$rank$\mathcal{B}_1 = 1$. Since here, $\mathcal{N}_2$ does not exist, rank$\mathcal{B}_1$ is equal to zero.
- β_0 =rank$\mathcal{Z}_0-$rank$\mathcal{B}_0$ = 3-2=1

13.2 Computing Persistent Homology

Compared to Homology which consider only one simplex, here we have a sequence of simplices; Filteration:

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$$
$$H_p(K_0) \to H_p(K_1) \to \cdots \to H_p(K_n)$$

For any $i \leq j$, inclusion map $K_i \to K_j$ induces another map, Homomorphism $f_p^{ij} : H_p(K_i) \to H_p(K_j)$.

Definition 13.1. *The p-th Persistent Homology groups are the images of the Homomorphism induces by inclusion.*

$$H_p^{ij} = im\ f_p^{ij} \qquad 0 \leq i \leq j \leq n$$

p-th persistent Betti number: $\beta_p^{ij} = \text{rank } H_p^{ij}$

Review

Definition 13.2. *A **Homomorphism** is a map between two algebraic structures of the same type that preserves the operation of the structures.*

$$f : A \to B \quad s.t.\ f(x+y) = f(x) + f(y) \quad \text{for every } x, y \in A$$

Here we say " f preserves the operation".

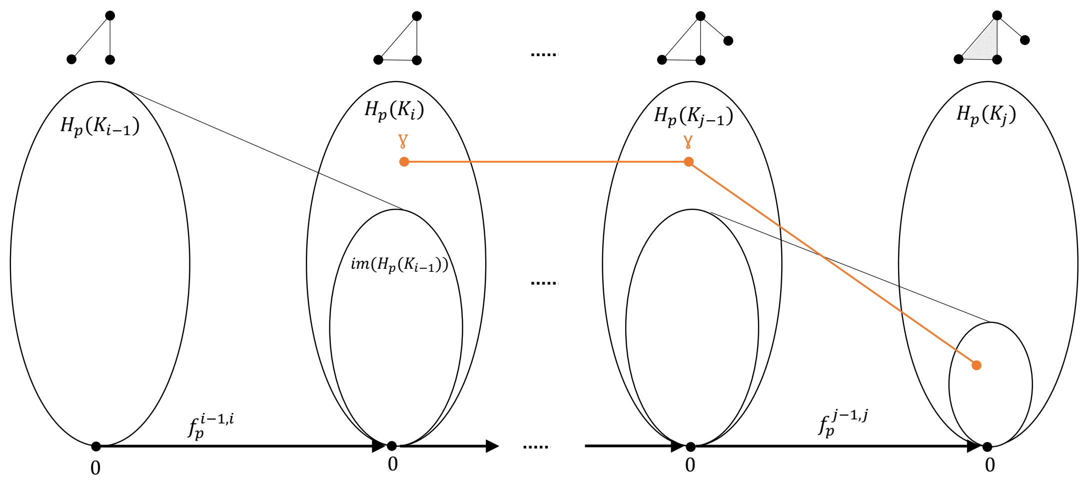


Figure 13.1:

We have: $H_p = \frac{Ker\ \delta_p}{im\ \delta_{p+1}}$. In Figure 13.1, we have the following observations:

1. γ born at K_i (not in the image of the $H_p(K_{i-1})$)
2. γ dies entering K_j (First time falls into *im*$(H_p(K_{j-1}))$)

In the example drawn on top of Figure 13.1, γ is the tunnel inside the empty triangle.

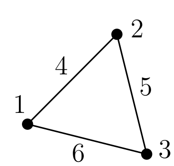
Let m be the number of simplicial complexes. We have the following algorithm to compute the reduced matrix from the boundary matrix:

$R = \delta_i$
for $j = 1$ to m **do**
  while $\exists j_0 < j$ with low$(j_0) =$ low(j) **do**
    add column j_0 to j
  end while
end for

Recall: for $i =$ low(j), i-th row contains the lowest 1 of the j-th column.

13.2.1 Example One

Consider the following simplicial complex which is an empty triangle. The appearance order of the simplices is shown in the Figure.



We apply the algorithm described in the previous section to compute Persistent Homology.

$$\partial = \begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & & & & 1 & & 1 \\ 2 & & & & 1 & 1 & \\ 3 & & & & & 1 & 1 \\ 4 & & & & & & \\ 5 & & & & & & \\ 6 & & & & & & \end{array}$$

$$\Rightarrow \begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6(+5+4) \\ \hline 1 & & & & 1 & & \not{1} \\ 2 & & & & \boxed{1} & 1 & \not{1} \\ 3 & & & & & \boxed{1} & \not{1} \\ 4 & & & & & & \\ 5 & & & & & & \\ 6 & & & & & & \end{array}$$

The reduced matrix has two non-zero columns:

- column 4: **Birth:** 2 (=$low(4)$), **Death:** 4
  This is a 0-dimension simplex (vertex).
- column 5: **Birth:** 3 (=$low(5)$), **Death:** 5
  This is a 0-dimension simplex (vertex).

We define **Persistence = Death - Birth**.
For this example, the persistence diagrams are shown in Figure 13.2. Note that the node in dimension one persistence diagram corresponds to the tunnel.

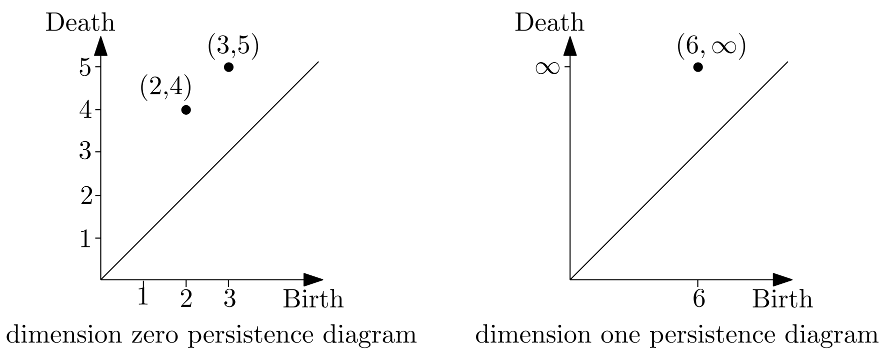


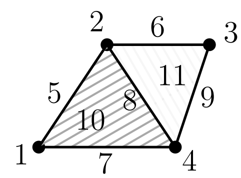
Figure 13.2:

Now, we look at the zero columns. This columns indicate births:

- A simplex is born at 2 (the column 2 is empty), and died at 4 ($low(4) = 2$)
- A simplex is born at 3 (the column 3 is empty), and died at 5 ($low(5) = 3$)
- A simplex is born at 1 (the column 1 is empty), and **never** dies ($\nexists i : low(i) = 1$). (Connected component)
- A simplex is born at 6 (the column 1 is empty), and **never** dies ($\nexists i : low(i) = 6$). (Tunnel)

13.2.2 Example Two

Consider the following simplicial complex as the second example:



$$
\partial = \begin{array}{c|ccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & & & & & 1 & & 1 & & & & \\
2 & & & & & 1 & 1 & & 1 & & & \\
3 & & & & & & 1 & & & 1 & & \\
4 & & & & & & & 1 & 1 & 1 & & \\
5 & & & & & & & & & & 1 & 1 \\
6 & & & & & & & & & & & \\
7 & & & & & & & & & & 1 & \\
8 & & & & & & & & & & 1 & 1 \\
9 & & & & & & & & & & & 1 \\
10 & & & & & & & & & & & \\
11 & & & & & & & & & & &
\end{array}
$$

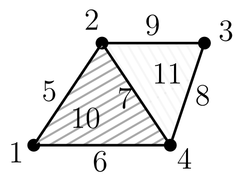
$$
\Rightarrow \begin{array}{c|ccccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8(+7+5) & 9 & 10 & 11 \\
1 & & & & & 1 & & 1 & \cancel{1} & \cancel{1} & & \\
2 & & & & & \boxed{1} & 1 & & \cancel{1} & \cancel{1} & & \\
3 & & & & & & \boxed{1} & & & \cancel{1} & & \\
4 & & & & & & & \boxed{1} & \cancel{1} & \cancel{1} & & \\
5 & & & & & & & & & & 1 & 1 \\
6 & & & & & & & & & & & \\
7 & & & & & & & & & & 1 & \\
8 & & & & & & & & & & \boxed{1} & 1 \\
9 & & & & & & & & & & & \boxed{1} \\
10 & & & & & & & & & & & \\
11 & & & & & & & & & & &
\end{array}
$$

Based on the reduced matrix, we have the following observations:

- (2,5), dimension zero simplex (edge)
- (3,6), dimension zero simplex (edge)
- (4,7), dimension zero simplex (edge)
- (8,10), dimension one simplex (tunnel)
- (9,11), dimension one simplex (tunnel)
- $(1,\infty)$, connected component

13.2.3 Example Three

Here, we have the same simplicial complex as in the previous example but with a different filtration (ordering).



$\delta =$

	1	2	3	4	5	6	7	8	9	10	11
1					1	1					
2					1		1		1		
3								1	1		
4						1	1	1			
5										1	
6										1	
7										1	1
8											1
9											1
10											
11											

$\Rightarrow$

	1	2	3	4	5	6	7(+6+5)	8(+6)	9(+8+5)	10	11
1					1	1	~~1~~	**1**	~~1~~		
2					$\boxed{1}$		~~1~~		~~1~~		
3								$\boxed{1}$	~~1~~		
4						$\boxed{1}$	~~1~~	~~1~~			
5										1	
6										1	
7										$\boxed{1}$	1
8											1
9											$\boxed{1}$
10											
11											

Based on the reduced matrix, we have the following observations:

- (2,5), dimension zero simplex (edge)
- (4,6), dimension zero simplex (edge)
- (3,8), dimension zero simplex (edge)
- (7,10), dimension one simplex (tunnel)
- (9,11), dimension one simplex (tunnel)
- (1,∞), connected component