CS 6170 Computational Topology: Topological Data Analysis **University of Utah**
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9.1 Introduction

Topological Data Analysis, as we know captures higher order interactions in the data. The unique features of TDA make it a promising bridge between topology and geometry.

In the last class we saw the concepts of simplicial complex, k-chain, boundary of k-chain, cycle, group, k^{th} homology group and Betti number.

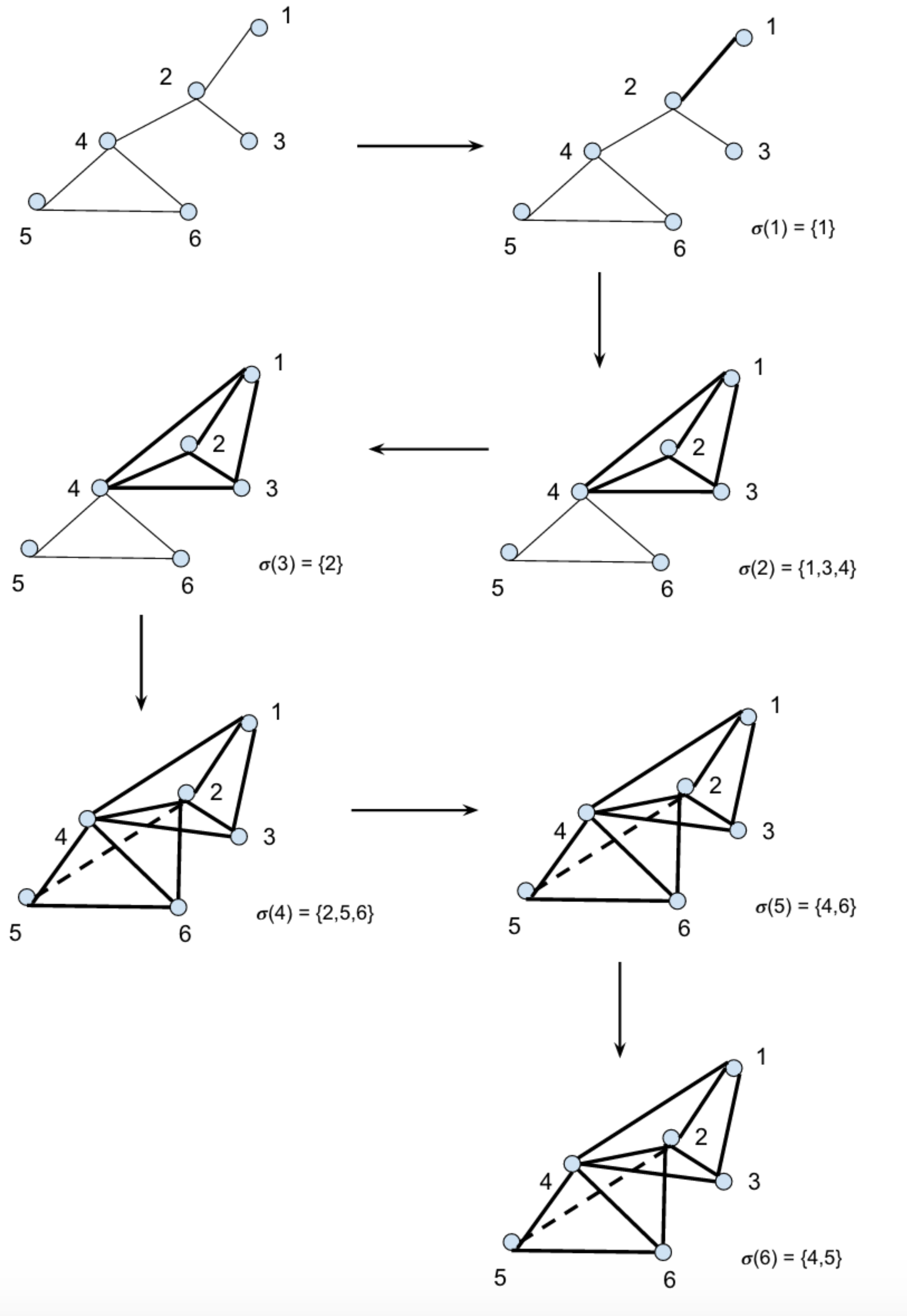
Today, we introduce persistence homology, in our lecture, which is a method for computing topological features of a space at different spatial resolutions.

9.2 Construction of Simplicial Complexes from Graphs

Simplicial complexes could be constructed from both undirected or directed graphs (digraphs). We now consider two ways to do this: the neighborhood complex and the clique complex.

9.2.1 Neighborhood complex

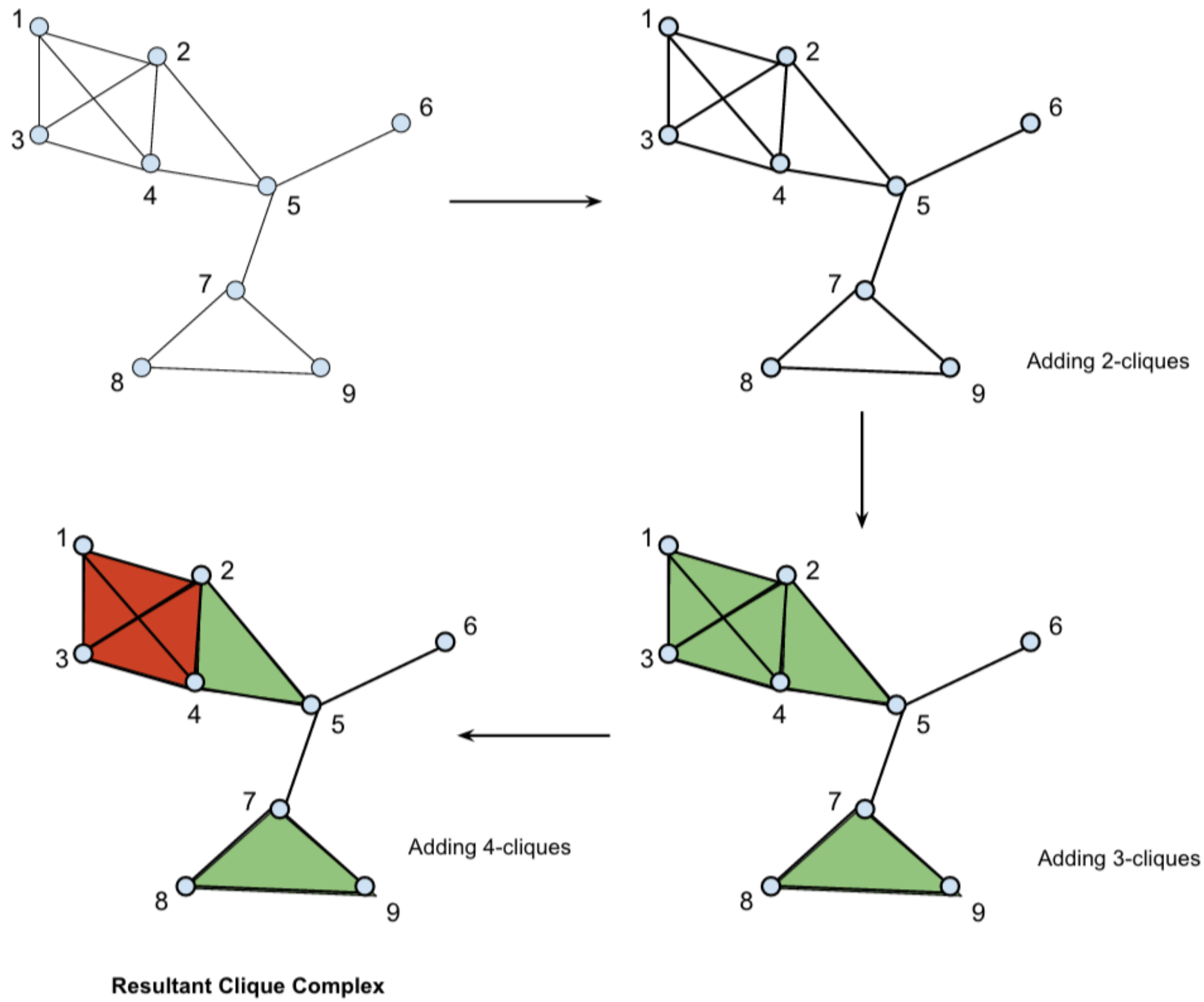
We can construct the neighborhood complex N(G), with vertices $v_1, ..., v_n$, from the graph G, in such a way that, for each vertex v of G, there is a simplex containing the vertex v along with its neighbouring vertices, corresponding to the directed edges $v \to w$.
To construct this, we can take each vertex in $v_1, ..., v_n$ one by one, and construct the simplex along with it's neighbors, each time.
The following figure gives an example on how to compute the neighborhood complex from a graph:



σ(1) = {1}
σ(2) = {1,3,4}
σ(3) = {2}
σ(4) = {2,5,6}
σ(5) = {4,6}
σ(6) = {4,5}

9.2.2 Clique complex

The clique complex C(G) has the complete subgraphs as simplices and the vertices of G as its vertices so that it is essentially the complete subgraph complex. The maximal simplices are given by the collection of vertices that make up the cliques of G.
The following figure gives an example on how to compute the clique complex from a graph:



9.3 Persistence Homology

9.3.1 Filteration

The basic aim of persistent homology is to measure the life-time of certain topological properties of a simplicial complex when simplices are added or removed from the complex.

For this, we start out with an empty set and keep adding the simplices to the complex. The sequence of simplices constructed in this process is known as filteration.
Formally, the filteration of a simplicial complex K is a sequence of complexes K_i, such that:
$\phi = K_0 \subset K_1 \subset ... \subset K_n = K$

Usually, two filteration constructions are considered:

1. The first one is formed when at each stage of the filtration only one simplex is added. An example of the same is given below:

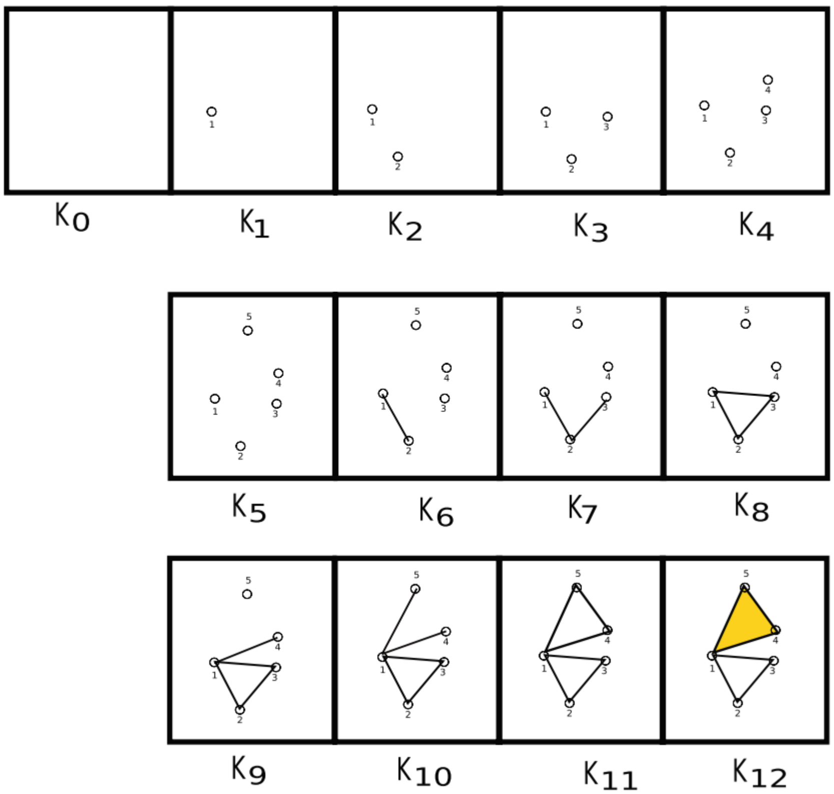
   

   Figure from [2]

2. In the second construction, a simplex is added to the sequance, say a subcomplex K_j, when all its faces are part of some $K_i (i \leq j)$. Hence, this construction does not require only one simplex to be added at each stage of the filteration. An example of the same is shown below:

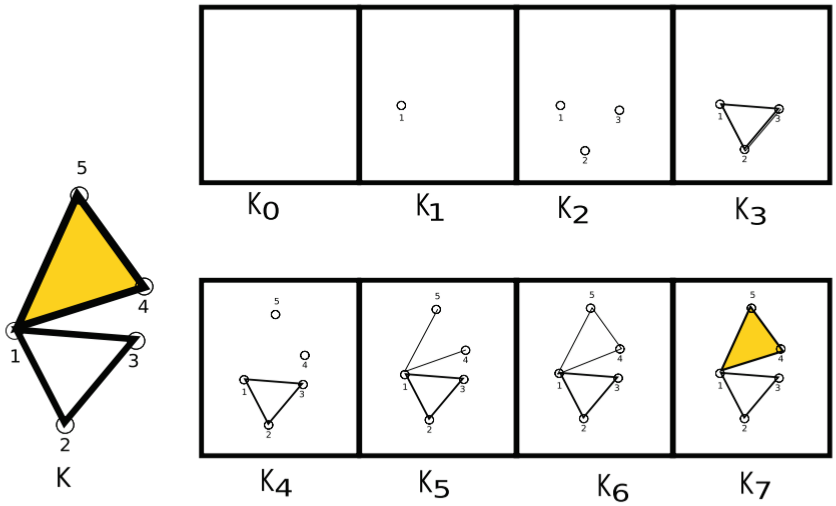
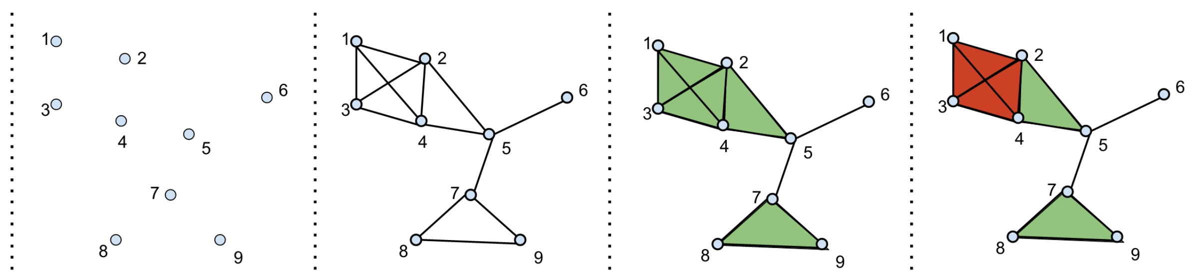


Figure from [2]

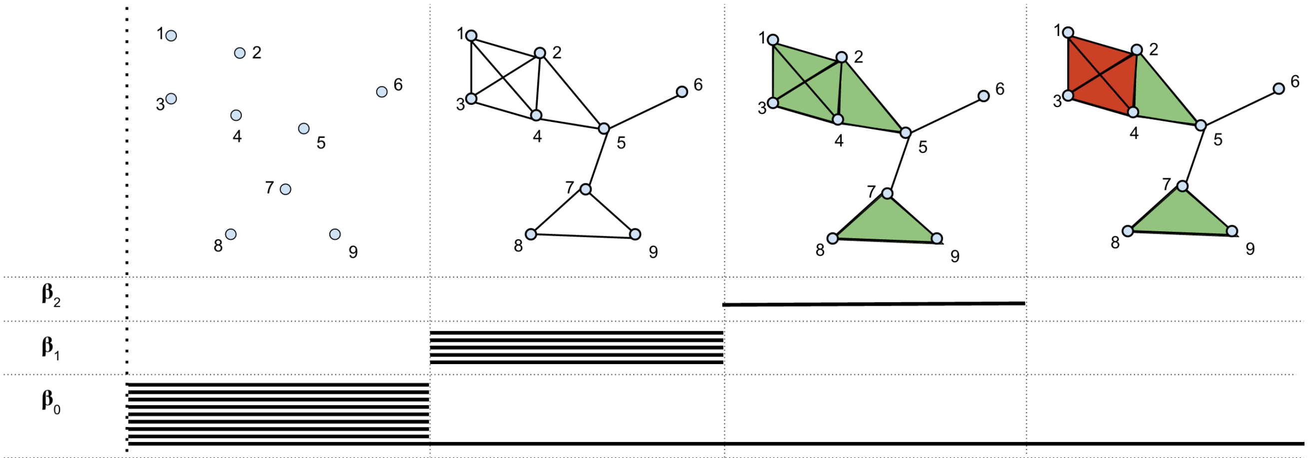
9.3.2 Visualization of persistence homology: Barcodes

Since persistent homology represents an algebraic invariant that detects the birth and death of each topological feature as the complex evolves in time, it is advantageous to encode the persistent homology in the form of a parametrized version of the rank of homology group i.e. its Betti number [2].
Here, we show how to do this, using Persistence Barcodes. To do this, let us take the example, we used to show the clique complex and calculate the Betti numbers for the same.



β_2	0	0	1	0
β_1	0	5	0	0
β_0	9	1	1	1

Now, we get the persistence barcode using the Betti numbers calculated above:


1
2
3
4
5
6
7
8
9
β2
β1
β0

References

[1] ”Topological Data Analysis”, from Wikipedia, *https://en.wikipedia.org/wiki/Topological_data_analysis*

[2] DANIJELA HORAK , SLOBODAN MALETI and MILAN RAJKOVI, “Persistent Homology of Complex Networks, 2008”, *http://www.sci.utah.edu/ beiwang/teaching/cs6170-spring-2017/HorakMaleticRajkovic_2008.pdf*