CS 6170 Computational Topology: Topological Data Analysis **University of Utah**
Spring 2017 **School of Computing**

Lecture 2: Feb 2, 2017

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2.1 Topics covered

- Algebraic topology
- Computational topology

2.1.1 Brief review of items from the previous lecture

- Simplicial complex K
- Underlying space of simplicial complex $\mid K \mid$ (piece of Euclidean space the simplicial complexes occupy)

2.1.2 Triangulation, Betti number

Definition: A *triangulation* of a topological space $\mathbb{X}$ is a simplicial complex K together with a homeomorphism between $\mathbb{X}$ and $\mid K \mid$.

Betti number: (also notated b_i)
β_0: counting the number of connected components
β_1: counting the number of tunnels
β_2: counting the number of voids
β_k: counting the number of higher-order voids

The Betti rank of some objects

Notation	Object	β_0	β_1	β_2
$\mathbb{S}^1$	circle	1	1	0
$\mathbb{S}^2$	sphere	1	0	1
$\mathbb{T}^2$	torus	1	2	1
$\mathbb{P}^2$	projective plane	1	0	0
$\mathbb{K}^2$	klein bottle	1	1	0
–	double torus	1	4	1
–	g("genus")-hole torus	1	$2 \times g$	1

Definition: The *genus* of a connected orientable surface is an integer representing the maximum number of cuttings possible along simple closed curves without disconnecting the resulting manifold.

Examples:

Notation	Object	Genus
$\mathbb{T}^2$	torus	1
–	double torus	2
$\mathbb{S}^2$	sphere	0
–	"pretzel"	3

Any further cuts would separate the manifolds into more than one piece.

2.1.3 Homology groups

Notation:

H_0	0-dimension homology group
H_1	1-dimension homology group

Groups and abelian groups
Definition: A group is a set G with an operation, such that:

1. $a, b \in G \Rightarrow a \cdot b \in G$ (closure)
2. $a, b, c \in G \Rightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity)
3. $\exists\, e \in G$ s.t. $\forall a \in G$, $e \cdot a = a \cdot e = a$ (identity element)
4. $\forall\, a \in G$, $\exists\, b \in G$, s.t. $a \cdot b = b \cdot a = e$

Additionally, for an abelian group:
$\forall\, a, b \in G$, $a \cdot b = b \cdot a$

Example 1: integers under the addition operation

1. $a, b \in \mathbb{Z} \Rightarrow a + b \in \mathbb{Z}$
2. $(a + b) + c \Rightarrow a + (b + c)$
3. $0 + a = a + 1 = a$; that is, $e = 0$
4. $(-a) + a = 0$

Example 2: group of integers, modulo 2, with addition (that is, $\mathbb{Z}^2$ or the set $\{0, 1\}$)

- $1 + 1 = 2 = 0$ (closure)
- $(1 + 1) + 1 = 1 + (1 + 1) = 3 = 1$ (associativity)
- $0 + 1 = 1$ (identity element, 0)
- $\forall\, a \in \{0, 1\}$, $a + a = e$ (inverse)
- $1 + 0 = 0 + 1 = 1$ (commutativity)

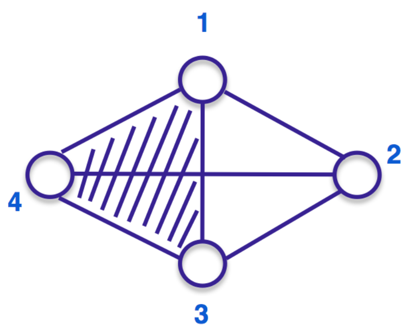
Homology
Let K by a simplicial complex and p a number of dimensions.

Definition: modulo 2 coefficient
$a \in \mathbb{Z}_2$, $a = 0$ or $a = 1$
This is also known as the $\mathbb{Z}_2$ coefficient.

Definition: A p-chain is a formal sum "+" of p-simplices in K.
$c = \sum a_i \sigma_i$

Here, $a_i = 0$ or 1 (in our context). Basically, just add them together.

For example:



$K = \{1, 2, 3, 4, 1\ 2, 2\ 4, 1\ 4, 2\ 3, 3\ 4, 1\ 2\ 4\}$

Summing 3 vertices would make a 0-dimension chain (or 0-chain):
$c_0 = 1 + 2 + 3$

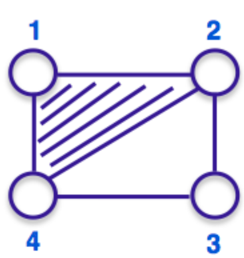
To make a 1-chain:
$c_1 = 1\ 2 + 2\ 3 + 3\ 4 + 4\ 1 + 0 * 2\ 4$
(The last term multiplied by zero can just be ignored.)

A 2-chain is a linear combination of 2-simplices.
$c_2 = 1\ 2\ 4$

An n-chain is a linear combination of n-simplices. That is, we use chain addition, done via component-wise addition.

$c = \sum a_i \sigma_i$ plus $c' = \sum b_i \sigma_i$ becomes $c + c' = \sum (a_i + b_i) \sigma_i$

Example:



$c_1 = 1\,2 + 2\,3 + 3\,4 + 1\,4$
$c_1' = 2\,3 + 3\,4 + 2\,4$
$c_1 + c_1' = 1\,2 + 2\,3 + 3\,4 + 1\,4 + 2\,3 + 3\,4 + 2\,4 = 1\,2 + 1\,4 + 2\,4$

Above, repeating numbers get cancelled out. The result is the boundary of the shaded triangle.

Definition: Chain group
The p-chain together with the addition operation from the group of p-chains.

Notation: $(c_p, +)$ or $c_p = c_p(K)$
$c_0 =$ 0-chain group
$c_1 =$ 1-chain group
$c_2 =$ 2-chain group

2.1.4 Homology

A homology group has the following attributes:

1. $c, c' \in C_p \Rightarrow c + c' \in C_p$
2. (c + c') + c" = c + (c' + c")
3. 0 + c = c + 0 = c
4. c + c = 0 (recall the mod 2 coefficient)

Definition: The *boundary* of a p-dimensional simplex is the sum of its (p-1)-dimensional face.

$\sigma = [u_0, ...u_p]$

$\partial_p \sigma = \sum_{j=0}^{p} [u_0, ...u_j ...u_p]$

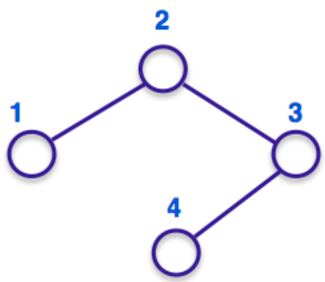
Example:

$\sigma = [1, 2, 3]$

For chain addition:
$\partial(c + c') = \partial c + \partial c'$

And for a chain $c = \sum a_i \sigma_i$,
$\partial = \sum a_i (\partial \sigma_i)$

Example:



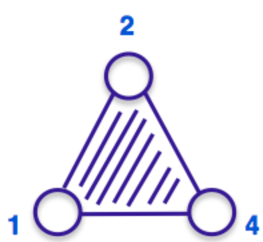
$c = 1\,2 + 2\,3 + 3\,4$

$\partial c = 1 + 2 + 2 + 3 + 3 + 4 = 1 + 4$

Example:

$c = 1\,2$
$\partial c = 1 + 2$

Example:



$c = 1\,2\,4$
$\partial c = 1\,2 + 2\,4 + 4\,1$ (all of its edges)

Applying a boundary to itself yields zero.
$\partial(\partial c) = 1 + 2 + 2 + 4 + 1 + 4 = 0$

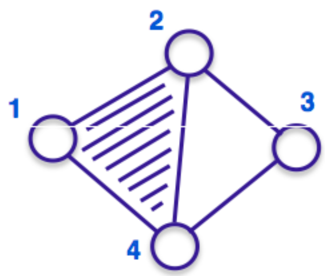
Definition: A p-cycle is a p-chain with an empty boundary.
$\partial c = 0$

A group of p-cycles is called a p-dimensional cycle group.
$Z_p = Z_p(K)$
(subgroup of $C_p(K)$)

Definition: A p-dimensional boundary is a p-chain that is the boundary of a (p+1)-dimensional chain.
$c = \partial d$ with $d \in c_{p+1}$

A group of p-boundary is denoted:
$B_p = B_p(K)$

Example:

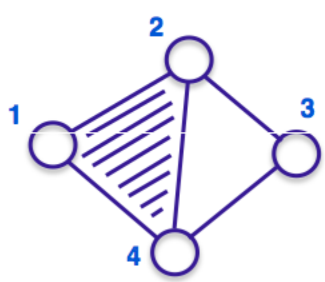


$c \in B_1(K)$
$c = 1\,2 + 2\,4 + 1\,4$
$c = \partial(1\,2\,4)$

2.1.5 Homology group

Definition: The p-th homology group is the p-th cycle group mod the p-th boundary group.
$H_p = Z_p/B_p$

Example:



$c = H_1$
$c = 2\,3 + 3\,4 + 2\,4$

Here, c is a 1-chian. It is not bounding a higher-dimensional thing, so it's not a boundary. It is a tunnel.

$c' = 1\,2 + 2\,3 + 3\,4 + 1\,4$
This is a 1-d chain. It's a cycle (closed loop) and not a boundary, because there is no 2-d chain it is bounding.