Apr 18

- Local Homology (LH)
- Persistent Local Homology (PLH)

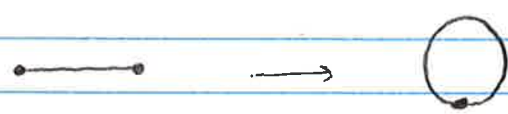
⇒ Applications : ① Root architecture
② Road networks
③ Stratification learning / clustering.

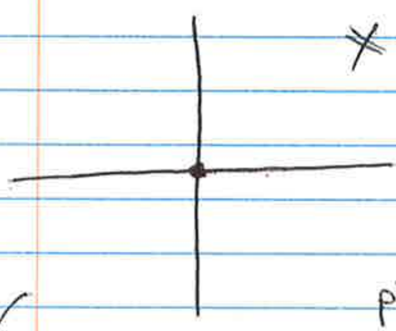
LH : $\mathbb{X}$: topological space, let $x_0 \in \mathbb{X}$ be a point in $\mathbb{X}$.

① $H_p(\mathbb{X}, \mathbb{X} - x_0)$: Relative homology of $\mathbb{X}$ w.r.t. $\mathbb{X} - x_0$
(relative homology of space w.r.t. the space minus the point.

② $\lim_{r \to 0} H_p(\mathbb{X}, \mathbb{X} \setminus U_r)$ where U_r : nbd of x_0 of radius r

③ $\lim_{r \to 0} H_p(\mathbb{X} \cap U_r, \mathbb{X} \cap \partial U_r)$

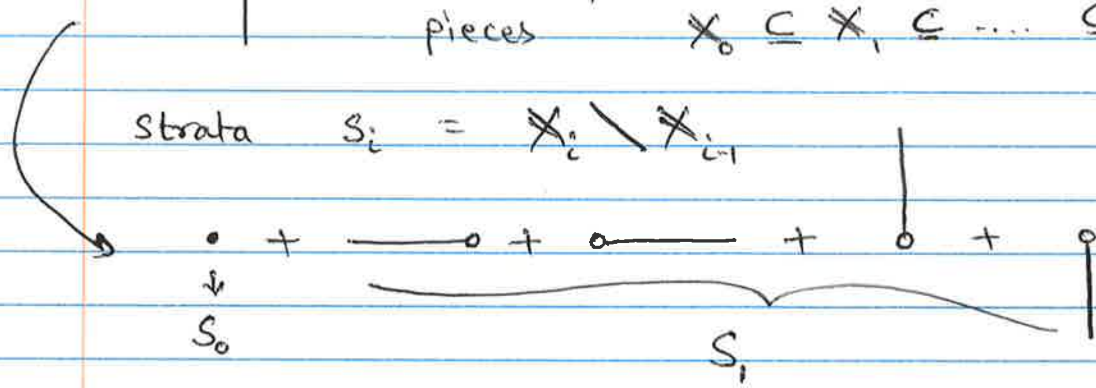
Step :  ⇒ rank $H_1(\quad) = 1$

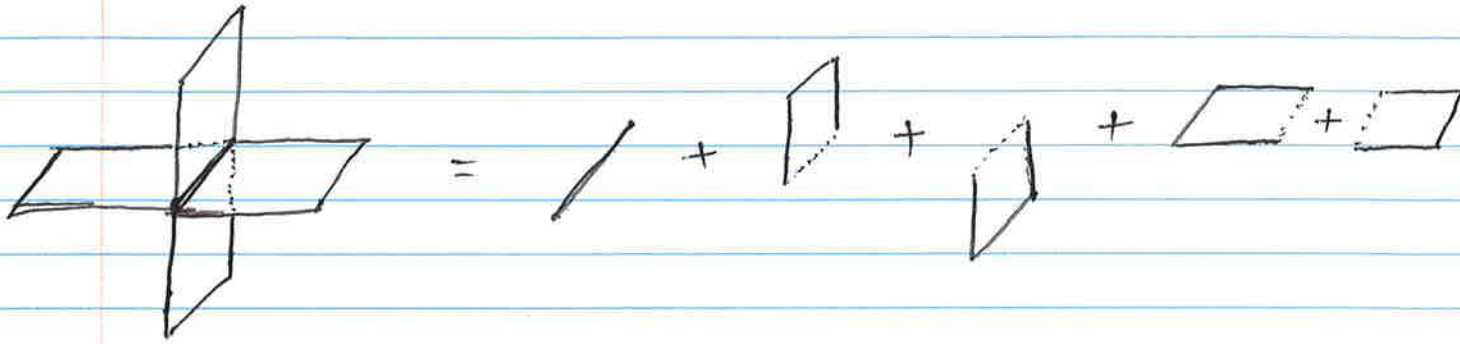


$\mathbb{X}$ ⇒ two 1-manifolds intersecting in a 0-dim point.

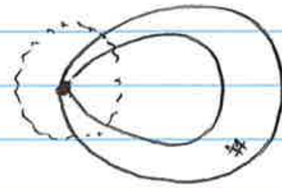
This is a stratified space.
i.e. $\mathbb{X}$ can be decomposed into manifold pieces $\mathbb{X}_0 \subseteq \mathbb{X}_1 \subseteq \dots \subseteq \mathbb{X}_d \subseteq \mathbb{X}$

Strata $S_i = \mathbb{X}_i \setminus \mathbb{X}_{i-1}$

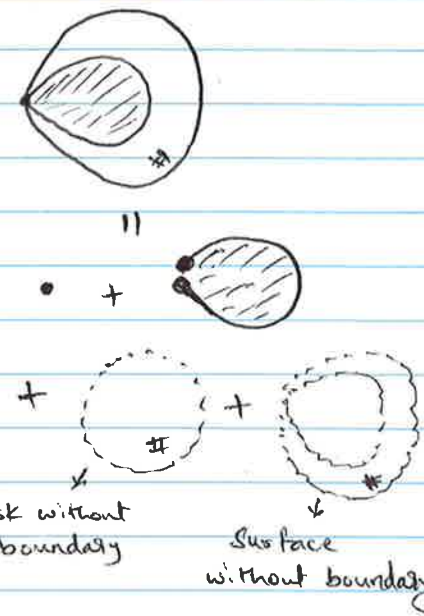




Torus



Pinched torus

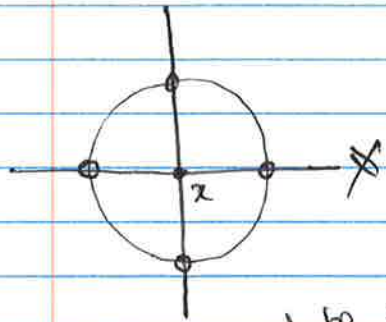
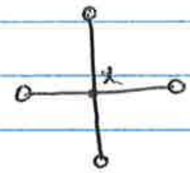


 ⇒ local neighborhood

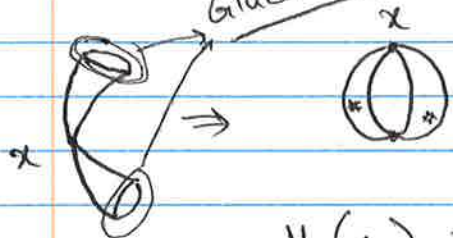
local neighborhood of

x : 

y : 

 $\Rightarrow$  $\Rightarrow$ 

Glued to a single point.



Glue all 4 boundary points into one.

$H_1(\;) = 3$

$H_0(\cdot) = 1$

$H_2(\cdot) = 2$

$H_1(\cdot) = 1$

$\rightarrow$ all three sheet boundaries glued into one point

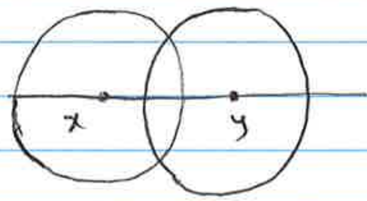
$\Rightarrow$ creates 2 voids

$\Rightarrow H_2(\cdot) = 2, \quad H_1(\cdot) = 0$

$\rightarrow$ We can use stratification learning as a pre-processing step and then apply manifold learning techniques.

manifold : locally, every point looks similar $\Rightarrow$ no strata
i.e. local homology will be same for all points.

local homology can describe complexity of local neighborhood.



① $H_1\left(\mathbb{X} \cap B_r(x),\ \mathbb{X} \cap \partial B_r(x)\right)$

rank 1

② $H_1\left(\mathbb{X} \cap B_r(y),\ \mathbb{X} \cap \partial B_r(y)\right)$

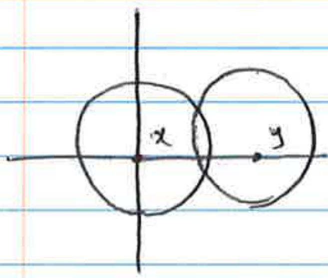
rank 1

③ $H_1\left(\underbrace{\mathbb{X} \cap B_r(x) \cap B_r(y)}_{}\ ,\ \mathbb{X} \cap \partial\left(B_r(x) \cap B_r(y)\right)\right)$

intersection of the two balls,

also has rank 1

There exist isomorphisms from ① to ③ and from ② to ③



① local homology @ x : rank $H_1 = 3$

② — " — @ y : rank $H_1 = 1$

③ — " — @ intersection of the two balls : rank $H_1 = 1$

⇒ There is no isomorphism

i.e. kernel of a map from ① to ③ is non-trivial.

3 loops mapping to 1 loop : 2 loops map to zero.

* local homology transfer maps

PLH : $H_p\left(\mathbb{X} \cap U_r,\ \mathbb{X} \cap \partial U_r\right)$

α. thickening of underlying space.

α - filtration : $H_p\left(\mathbb{X}_\alpha \cap U_r,\ \mathbb{X}_\alpha \cap \partial U_r\right)$

$$\downarrow$$

α' : $H_p\left(\mathbb{X}_{\alpha'} \cap U_r,\ \mathbb{X}_{\alpha'} \cap \partial U_r\right)$

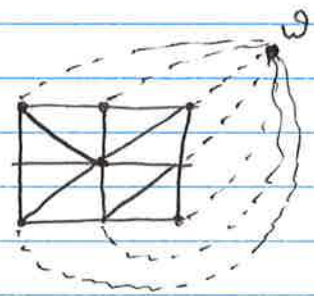
$\alpha \subseteq \alpha'$

γ- filtration : $H_p(X \cap U_\gamma, X \cap \partial U_\gamma)$

$$\downarrow$$

γ'- ~~B~~ $H_p(X \cap U_{\gamma'}, X \cap \partial U_{\gamma'})$

$\gamma \leq \gamma'$



Create a dummy vertex ω then connect all vertices on the boundary to it and also create triangles.