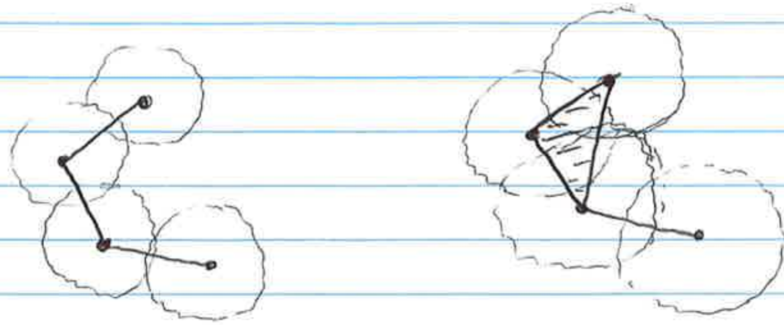
Mar 30

Mapper

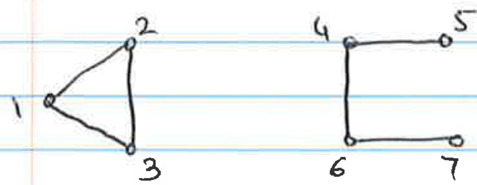
Maps between covers



$K_1 \subseteq K_2$

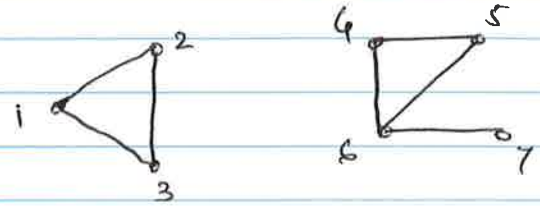
Def: Simplicial Map: Let K, L be two finite S.C over vertex set V_K and V_L. A set map $\phi : V_K \to V_L$ is a simplicial map if $\phi(\sigma) \in L$ for all $\sigma \in K$

K L



12 → 45
13 → 46
23 → ?

} Not well defined



K L

12 → 45   13 → 46   23 → 56

well defined mapping.

∴ Simplicial map.

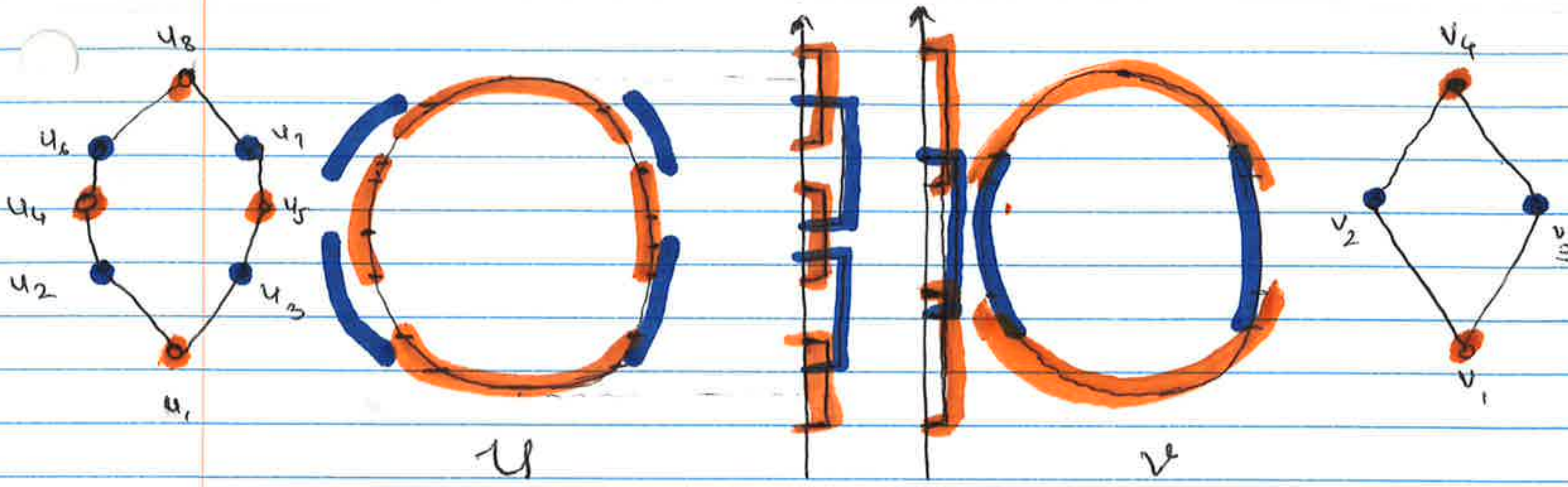
Def: If we have two covers of X, $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$

A map of covers from $\mathcal{U}$ to $\mathcal{V}$ $\mathcal{V} = \{V_\beta\}_{\beta \in B}$

is a set map $\Gamma : A \to B$

so that $U_\alpha \subseteq V_{\Gamma(\alpha)}$ for all $\alpha \in A$

Given such a map of covers, there is an induced simplicial map $\Gamma^* : N(\mathcal{U}) \to N(\mathcal{V})$ given on vertices by Γ



$u_1 \to v_1$ $\quad$ $u_4 \to v_2$ $\quad$ $u_6 \to v_4$

$u_2 \to v_1$ $\quad$ $u_5 \to v_3$ $\quad$ $u_7 \to v_4$

$u_3 \to v_1$ $\quad$ $u_8 \to v_4$

mapping between covers.

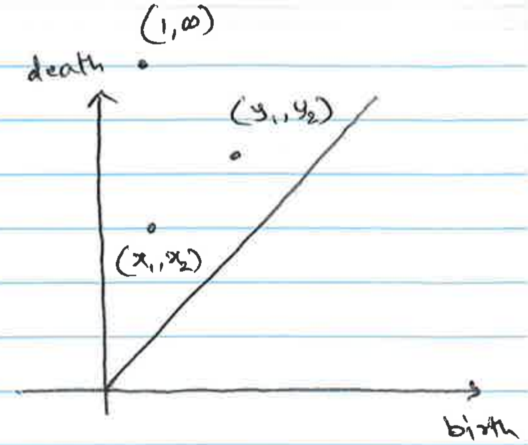
→ for mapping between corresponding S.C.c (Reeb graphs) vertices map the same way.

The edges u_1u_2 and u_1u_3 shrink, u_8u_6 and u_8u_7 also shrink (both end-points map to same vertex in V)

$u_2u_4 \to v_1v_2$, $\quad$ $u_3u_5 \to v_1v_3$

$u_4u_6 \to v_2v_4$ $\quad$ $u_5u_7 \to v_3v_4$

Stability



Persistent diagram : multi-set of points in the extended plane

$$\bar{\mathbb{R}}^2 = (\mathbb{R} \cup \{-\infty, +\infty\})^2$$

It contains finite number of off-diagonal points and infinite number of points on the diagonal

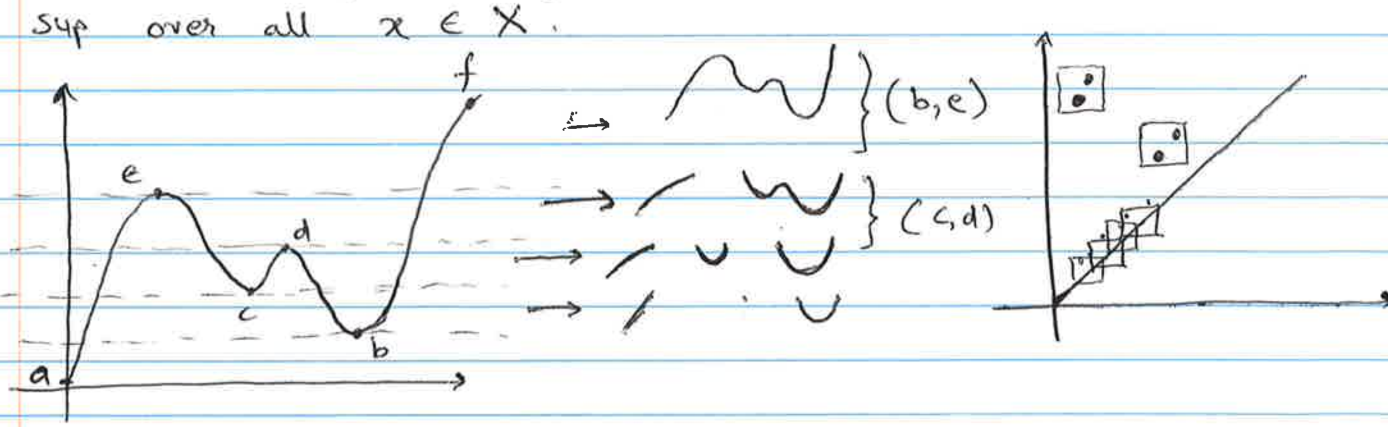
Let $x = (x_1, x_2)$, $y = (y_1, y_2)$

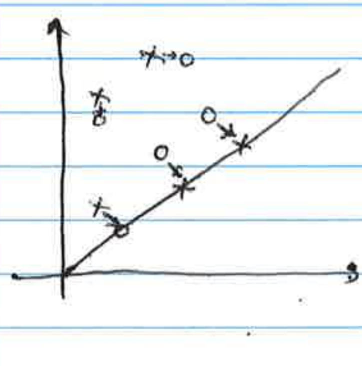
def : L_∞ norm : $\|x - y\|_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

Def : Bottleneck Distance : Let X, Y be two persistent diagrams with $\eta : X \to Y$ a bijection then the bottleneck distance

$$W_\infty(X, Y) = \inf_{\eta: X \to Y} \sup_{x \in X} \|x - \eta(x)\|_\infty$$

inf over all possible bijections
sup over all $x \in X$.





Thm: Stability of tame (well-behaved) function

Let $\mathbb{X}$ be a triangulable topological space and $f, g : \mathbb{X} \to \mathbb{R}$ be two tame functions.

for each dimension p,

$$W_\infty\left(Dgm_p(f),\ Dgm_p(g)\right) \leq \| f - g \|_\infty$$

→ if two functions are "close", their persistent diagrams are "close"

→ it is possible to have identical persistent diagrams even though functions are not close eg. if f mirrors g

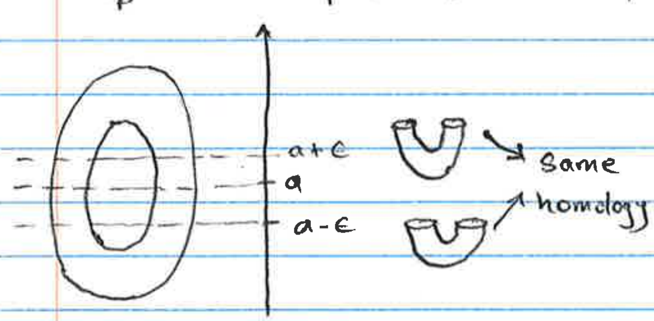
\# W_∞ is a metric!

1. $W_\infty(X, Y) = 0$ iff $X = Y$
2. $W_\infty(X, Y) = W_\infty(Y, X)$
3. $W_\infty(X, Z) \leq W_\infty(X, Y) + W_\infty(Y, Z)$

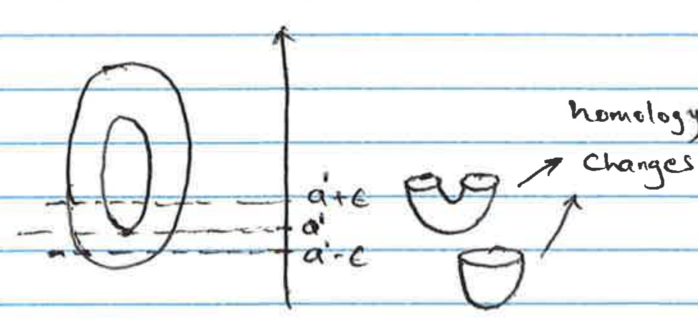
Def: Tame function: A function $f : \mathbb{X} \to \mathbb{R}$ is tame if it has a finite number of homological critical values and the homology groups of all sub-level sets have finite rank (implies finite number of off diagonal points)

Def: A point $a \in \mathbb{R}$ is a homological critical value if there is no $\epsilon > 0$ for which $f_p^{a-\epsilon, a+\epsilon}$ is an isomorphism for each p

$$f_p^{a,b} : H_p(\mathbb{X}_a) \to H_p(\mathbb{X}_b), \quad \mathbb{X}_a = f^{-1}(-\infty, a]$$



a is not h.c.v.

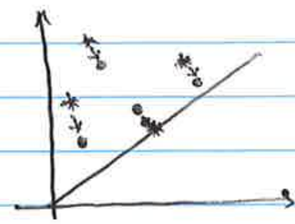


a' is h.c.v.

Def: degree q Wasserstein distance betⁿ two persistence diagrams

$$W_q(X, Y) = \left[\inf_{\eta : X \to Y} \sum_{x \in X} \| x - \eta(x) \|_\infty^q \right]^{1/q}$$

"transportation problem"
minimizes cost of moving or transporting all * points to corresponding • points



correspondence is given by bijection η. Minimize over all possible η

Stability:

$$W_q(Dgm_p(f), Dgm_p(g)) \leq C \cdot \| f - g \|_\infty^{1 - \frac{k}{q}} \quad \text{for } q \geq k > j$$

C and k are constants. $f, g : \mathbb{X} \to \mathbb{R}$ are tame, Lipschitz functions on metric spaces whose triangulations grow polynomially with constant exponent j

$\exists$ constants C, j s.t. $N(r) \leq C / r^j$

K : S.C $N(r)$: # simplices with max diameter at most r