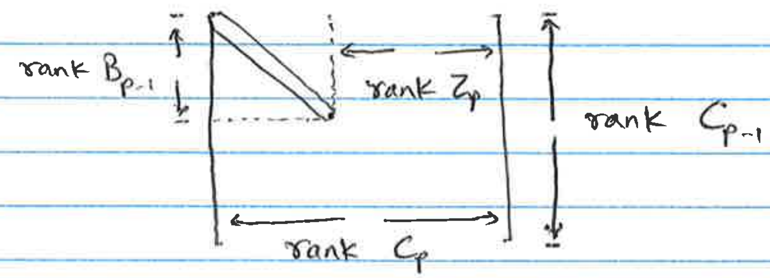
Feb 21

Review :

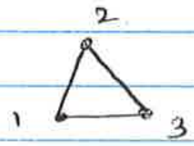
1) Computing homology : ∂_p : p-th boundary matrix $\xrightarrow{\text{reduce}}$ SNF (Smith Normal Form)

N_p :



β_p = rank Z_p − rank B_p

example :



$\beta_0 = 1$ $\tilde{\beta}_0 = \beta_0 - 1 = 0$

$\beta_1 = 1$ $\tilde{\beta}_1 = \beta_1 = 1$

homology reduced homology

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z}_2$$

$$\partial_0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \end{array} \qquad N_0 = \begin{array}{ccc} 1 & 2 & 3 \\ \hline 1 & 0 & 0 \end{array}$$

$$\partial_1 = \begin{array}{c|ccc} & 12 & 23 & 13 \\ \hline 1 & \boxed{1} & & X \\ 2 & X & \boxed{1} & X \\ 3 & & X & X \end{array} \qquad N_1 = \begin{array}{c|ccc} & 12 & 23 & 13 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \end{array}$$

From N_0 : rank $Z_0 = 2$ $\therefore \tilde{\beta}_0$ = rank Z_0 − rank $B_0 = 0$

From N_1 : rank $B_0 = 2$

rank $Z_1 = 1$ $\therefore \tilde{\beta}_1$ = rank Z_1 − $\underbrace{\text{rank } B_1}_{0}$ = 1

$$\beta_0 = \underbrace{\text{rank } Z_0}_{3} - \underbrace{\text{rank } B_0}_{2} = 1$$

3 → # Columns in ∂_1 or N_1

2. Computing Persistent homology (PH)

filtration: $K_0 = \phi \subseteq K_1 \subseteq \dots \subseteq K_n = K$

$$H_p(K_0) \longrightarrow H_p(K_1) \rightarrow \dots \rightarrow H_p(K_n)$$

- filtration: for $i \leq j$ inclusion map $K_i \longrightarrow K_j$
- induced homomorphism: $f_p^{ij} : H_p(K_i) \longrightarrow H_p(K_j)$

Def: The p-th PH group is the image of the homomorphism induced by inclusion:

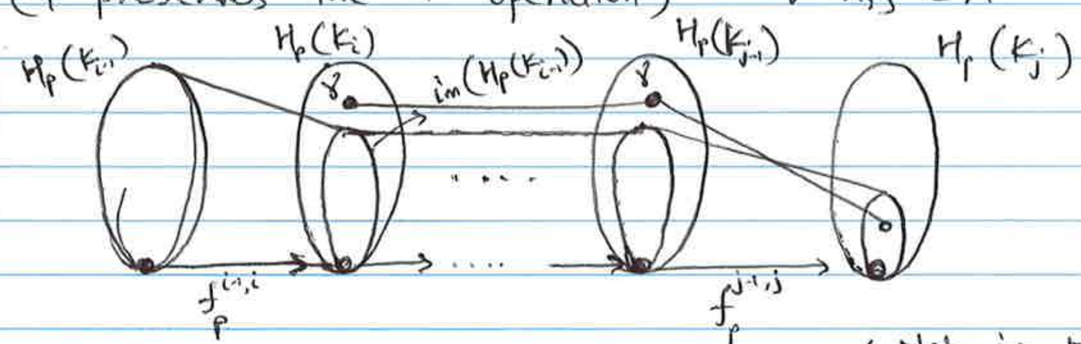
$$H_p^{ij} = \text{im } f_p^{ij} \qquad 0 \leq i \leq j \leq n$$

p-th ~~pers~~ persistent Betti number $\beta_p^{ij} = \text{rank } H_p^{ij}$

Def: Homomorphism is a map between two Algebraic structures of the same type, that preserves the operations of the two structures.

$$f: A \longrightarrow B, \quad f(x+y) = f(x) + f(y)$$

(f preserves the "+" operation) $\forall\, x, y \in A$

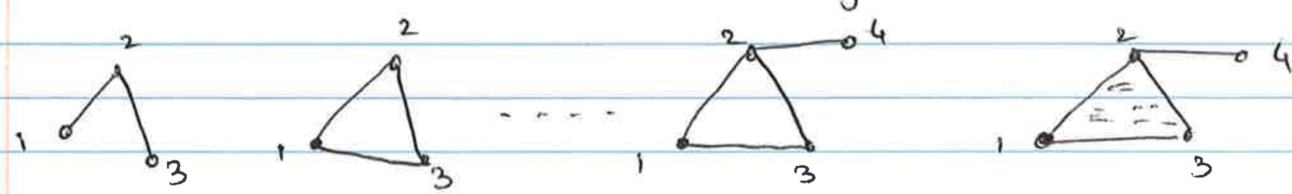


$$H_p = \frac{\ker \partial_p}{\text{im } \partial_{p+1}}$$

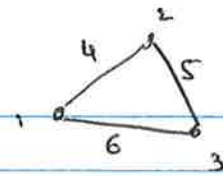
① γ born at K_i (Not in the image of $H_p(K_{i-1})$)

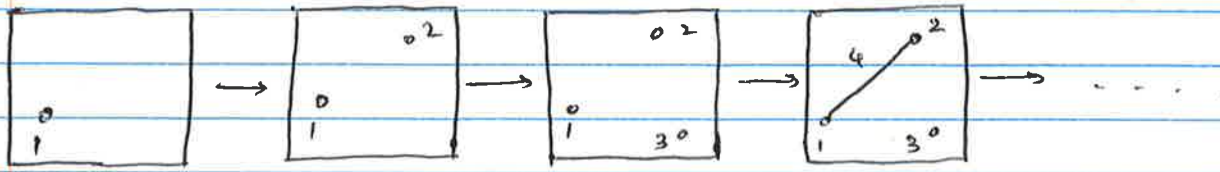
② γ dies entering K_j: in the image of $H_p(K_{j-1})$

↳ falls in the image for the first time



Computing PH example:





Algorithm:

$R = \partial_i$

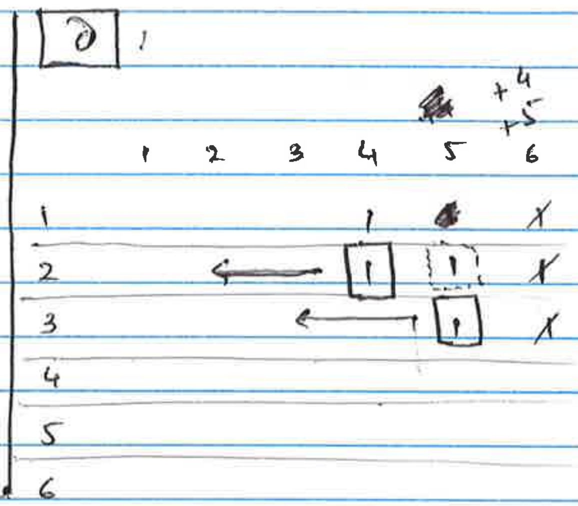
for $j = 1$ to m do   (m = # of S.C.)

  while $\exists\ j_0 < j$ s.t. $low(j_0) = low(j)$ do

    add col j_0 to col j

  end while

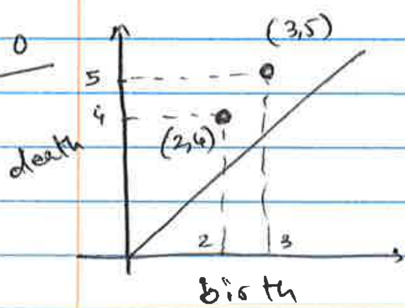
end for



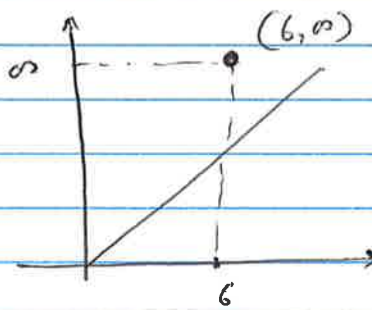
$i = low(j)$ if i^{th} row contains lowest 1 of j^{th} column.

→ After reduction: 2 non-zero columns ⇒ points (2,4), (3,5)

dim 0



dim 1



→ 4 zero columns: 1, 2, 3, 6.

2, 3 have lowest 1's i.e. something born at 2 (3) dies at 4 (5)

(1,∞)

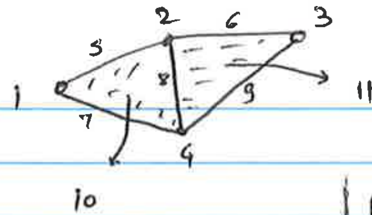
→ Something born at 1 → never dies → connected component

Something born at 6 → never dies → loop (6,∞)

∵ 1 is a vertex (dim 0) (1,∞) is a dim 0 feature

∵ 6 is an edge (dim 1) (6,∞) is a dim 1 feature

Example:



dim 0:
- ① $(2,5)$
- ② $(3,6)$
- ③ $(4,7)$

dim 1:
- ④ $(8,10)$
- ⑤ $(9,11)$

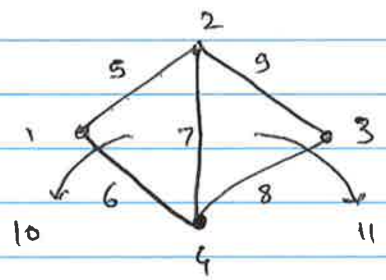
⑥ $(1,\infty) \to$ dim 0

6 Zero-columns: 1, 2, 3, 4, 8, 9

Only 1 has no low $\boxed{1}$ in row 1

								+5 +7	+5 +6 +7		
	1	2	3	4	5	6	7	8	9	10	11
1					1		1	~~1~~	~~1~~		
2					$\boxed{1}$	1		~~1~~	~~1~~		
3						$\boxed{1}$			~~1~~		
4							$\boxed{1}$	~~1~~	~~1~~		
5										1	
6											1
7										1	
8										$\boxed{1}$	1
9											$\boxed{1}$
10											
11											

example:



dim 0:
- $(2,5)$
- $(4,6)$
- $(3,8)$

dim 1:
- $(7,10)$
- $(9,11)$

$(1,\infty) \to$ dim 0

6 Zero-columns

1, 2, 3, 4, 7, 9

Only 1 has no low $\boxed{1}$ in corresponding row.

							+5 +6	+6	+7 +8		
	1	2	3	4	5	6	7	8	9	10	11
1					1	1	~~1~~	1	~~1~~		
2					$\boxed{1}$		~~1~~		~~1~~		
3								$\boxed{1}$	~~1~~		
4						$\boxed{1}$	~~1~~	~~1~~			
5										1	
6										1	
7										$\boxed{1}$	1
8											1
9											$\boxed{1}$
10											
11											