Feb 7

→ We use simplicial complexes as combinatorial representation of underlying topological space → we can do computations on simplicial complexes.

→ Last class: Given a simplicial complex K, we define:

(a) k-chain: sum of k-simplices in K

(b) boundary of a k-chain: sum of all $k-1$-dimensional faces.

(c) k-boundary: k-chain that is boundary of some $k+1$-chain.

(d) k-cycle: k-chain that has boundary $= 0$.

\# Group: Algebraic object: a set G with operation "+" that satisfies following properties:

i) Closure: $a, b \in G$ then $a+b \in G$.

ii) Associativity: $a+(b+c) = (a+b)+c$

iii) identity: $\exists\, e \in G$ s.t. $a+e = e+a = a$ $\forall\, a \in G$.

iv) inverse: $\forall\, a \in G$ $\exists\, b \in G$ s.t. $a+b = b+a = e$.

→ In addition, if (v) Commutative: $a+b = b+a$ then the group is called Abelian group.

→ With (addition modulo 2) [coefficients in $\mathbb{Z}_2 = \{0,1\}$]

C_k: Set of all k-chains is a group.

Z_k: Set of all k-cycles is a group: Subgroup of C_k

B_k: Set of all k-boundaries is a group: Subgroup of C_k

→ Boundary operator $\partial_k : C_k \to C_{k-1}$ [maps k-chains to their $k-1$ boundary]

$$\ker \partial_k = \{ z \in C_k \mid \partial_k(z) = 0 \} = Z_k$$

all ~~[illegible]~~ k-chains in C_k that map to 0 under ∂_k ⟶ all k-cycles.

$$\operatorname{im} \partial_k = \{ b \in C_{k-1} \mid b = \partial_k(z) \text{ for some } z \in C_k \} = B_k$$

k^{th} homology group of simplicial complex K is

$$H_k = Z_k / B_k = \ker \partial_k / \operatorname{im} \partial_{k+1}$$

[Cycles that are Not boundaries]

→ The quotient induces partitioning on Z_K. Each portion is an equivalence class.

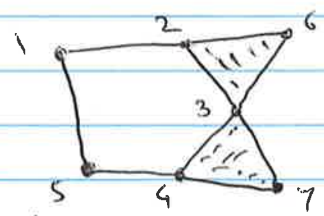
for two cycles z_1, z_2 in Z_K. if we can write

$z_1 = z_2 + b$ for some k-boundary $b \in B_K$ then

z_1, z_2 belong to same equivalence class.

→ Set of all partitions / equivalence classes forms a group $\Rightarrow H_k$.

→ Example :



$23+36+26 = [2,3,6]$, $34+47+37 = [3,4,7] \in B_1$

$\because$ boundaries of triangles.

1-chains : $[1,2,3,4,5]$ and $[1,2,6,3,4,5]$ are in same equivalence class

$(12+23+34+45+15)$

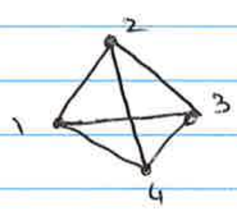
$+ (23+26+36)$

$= (12+26+36+34+45+15)$

$\therefore z_1 + b = z_2 \Rightarrow$ equivalent 1-cycles.

→ Basis of a group : smallest subset such that all the elements of the group can be represented as a sum of elements from the subset.

eg.



→ vertices and edges of tetrahedron.

1-cycles : ① $(12+23+13)$ ② $(12+24+14)$

③ $(23+34+24)$ ④ $(13+34+14)$

⑤ $(12+23+34+14)$

→ Pick any 3 cycles → all 5 cycles can be presented as a sum of those three.

ie. ④ = ① + ② + ③

⑤ = ② + ③

eg pick ① ② ③. then

$(12+23+34+14) = (12+24+14) + (23+34+24)$

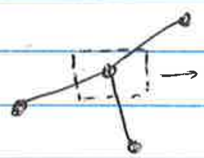
$(13+34+14) = (12+23+13) + (12+24+14) + (23+34+24)$

Betti Numbers: $\beta_k = \text{rank } H_k$ [# elements in the basis of H_k]

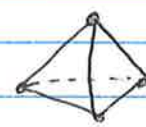
β_0: Connected Components
β_1: loops / tunnels → non bounding cycles.
β_2: voids

Simplicial Complexes from networks

① Neighborhood Complex ($N(G)$): For given graph G, for every vertex in G, include the simplex formed by the vertex and all its neighbors in $N(G)$

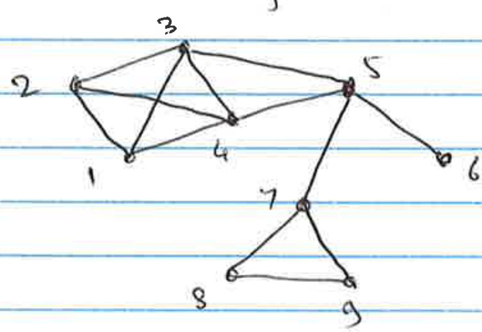


→ for this vertex →

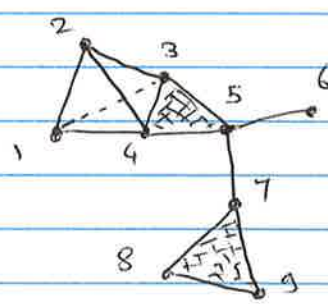


include tetrahedron formed by vertex and its 3 neighbors [3-simplex]

② Clique Complex: For every n-clique in G (complete subgraph of n vertices) include the n-1 simplex formed by the n vertices in clique complex $C(G)$



⟹



→ (1,2,3,4) form a 4-clique ∴ include 3-simlex (tetrahedron)
(7,8,9) and (3,4,5) form 3-cliques → include triangles

→ Mathematical representation of SCs:
in matrix form where columns are labeled by vertices and rows by simplexes in SC
$(i,j)^{th}$ element $= 1$ if simplex i has vertex j
0 otherwise.

Persistent Homology:

→ Real world data is noisy. it can induce ~~at~~ artifacts i.e. induce features that are not part of underlying space but are a product of noise in measurement etc.

→ We want to be able to distinguish essential features from artifacts.

→ Persistent homology: measure persistence / life time of topological features of simplicial complex as the SC grows → as simplexes are added to the SC.

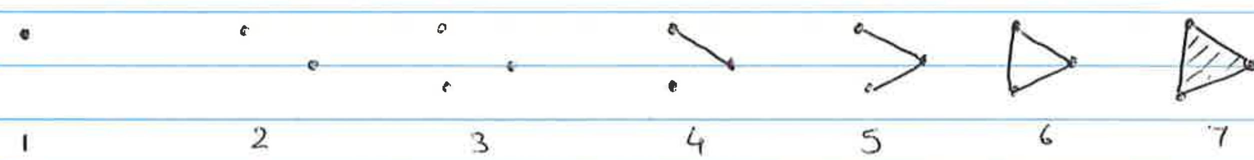
filtration: Sequence of simplicial complexes K_i such that

$$\phi = K_0 \subset K_1 \subset K_2 \ldots \subset K_n = K$$

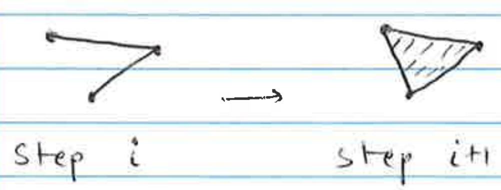
index i → rank of SC in filtration.
each K_i is a sub complex of K_{i+j} $\forall j \geq 1$..

→ Two ways of adding simplexes / defining filtration

① Add exactly one simplex at ~~a~~ each step:



② Add simplex as soon as all its faces are included



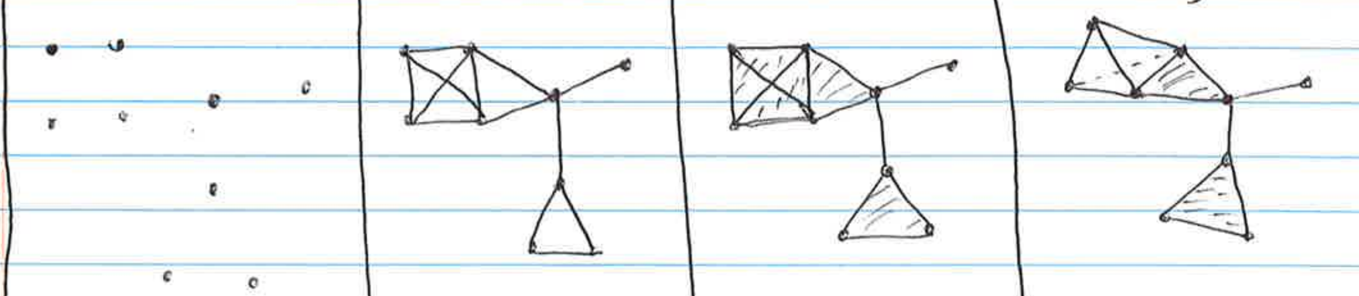
As soon as we add the third edge, all faces of the triangle are included so we also add the triangle.

Filtration on $C(G)$: Clique complex of G

$$K_i = \sum_{j=1}^{i} S_j$$

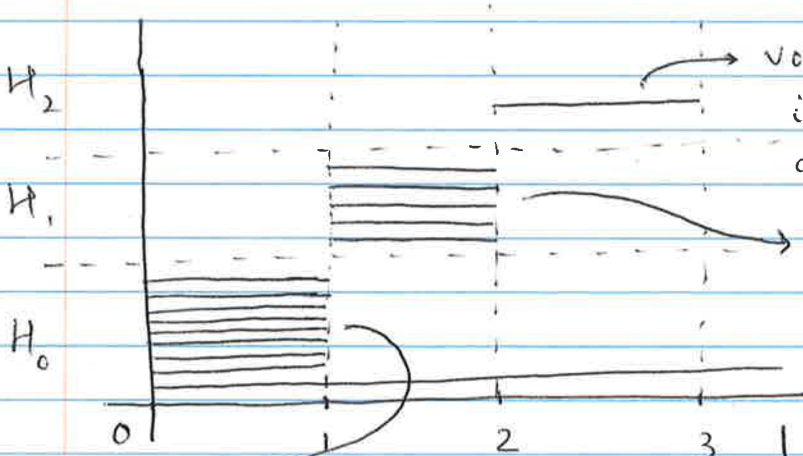
where S_j is the j-th skeleton of $C(G)$

i.e. K_i includes all simplices up to dimension i



K_0	K_1	K_2	K_3
Add vertices	Add edges	Add faces/triangles ↓ only faces of the tetrahedron included	Add tetrahedron entire tetrahedron included (volume filled in)

β_0	9	1	1	1
β_1	0	5	0	0
β_2	0	0	1	0



void formed in K_2 when we include all faces of tetrahedron. destroyed in step 3

Cycles formed in K_1 when we add edges in step 1

Only 5 bars because edges of tetrahedron have 3 cycles in basis. All cycles destroyed when we add faces in step 2.

We start with 9 vertices. each is a distinct component. in step 2 all are connected so only 1 bar survives.