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Supplemental Material: Visualizing Robustness of Critical Points for 2D Time-Varying Vector Fields

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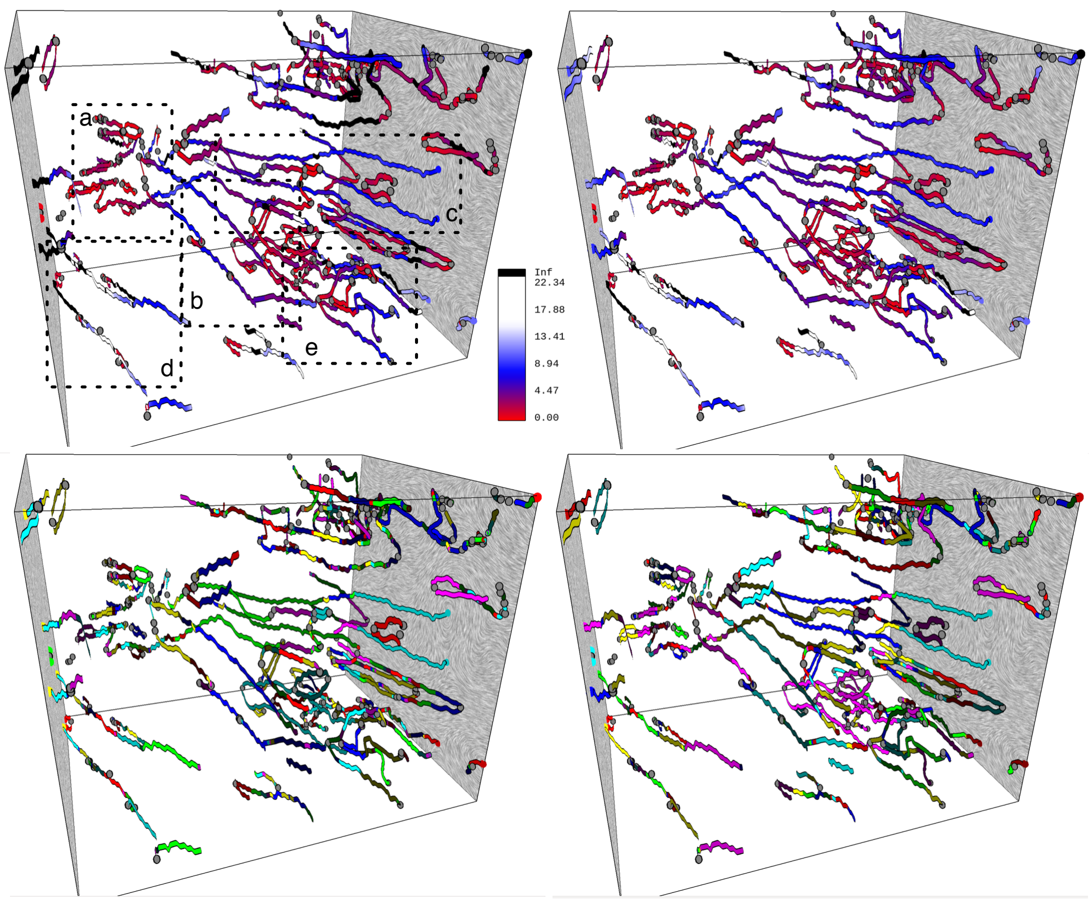

Figure 1: CentralAtlantic. *Top: robustness assignment along critical paths, for static (left) and dynamic (right) robustness. Bottom: robustness partners colored by unique values showcasing partner switches, for static (left) and dynamic (right) robustness. Local features are highlighted from a to e.*

1. Proofs Involving Static Robustness

The proofs of Lemma 3.1 and 3.2 are direct consequences of the following corollaries, whose proofs are based on the algebraic formulations within the proof of the Equivalence Theorem [CPS12]. We include their proofs here for completeness.

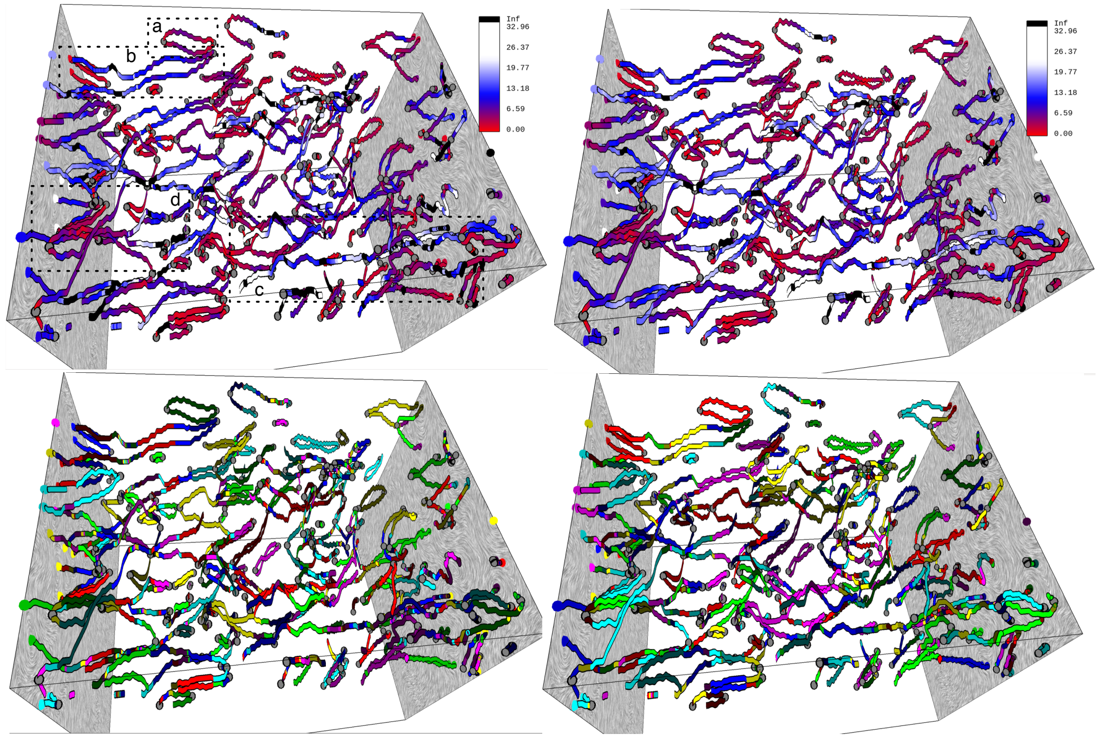

Figure 2: SouthAtlantic. *Top: robustness assignment along critical paths, for static (left) and dynamic (right) robustness. Bottom: robustness partners colored by unique values showcasing partner switches, for static (left) and dynamic (right) robustness. Local features are marked from a to d.*

Before we state the proofs of Corollary 1.1 and 1.2, we require a bit more algebraic understanding of the degree theory [CPS12]. Recall $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous vector field. Let r be a regular value of f_0. Let $C \subseteq \mathbb{F}_r$ be a path-connected component of $\mathbb{F}_r$. Function f restricted to C, denoted $f|_C : (C, \partial C) \to (B_r, \partial B_r)$, maps C to the closed ball B_r of radius r centered at the origin, where ∂ is the boundary operator. $f|_C$ induces a homomorphism on the homology level, $f_*|_C : \mathsf{H}(C, \partial C) \to \mathsf{H}(B_r, \partial B_r)$. Let μ_C and μ_{B_r} be the generators of $\mathsf{H}(C, \partial C)$ and $\mathsf{H}(Br, \partial B_r)$ respectively. The *degree* of C (more precisely the degree of $f|_C$), $\deg(C) = \deg(f|_C)$, is the unique integer such that $f_*|_C(\mu_C) = \deg(C) \cdot \mu_{B_r}$. Furthermore we have the function restricted to the boundary, that is, $f|_{\partial C} : \partial C \to \mathbb{S}^1$. It was shown that $\deg(f|_C) = \deg(f|_{\partial C})$ ([CPS12], Lemma 1).

Corollary 1.1 (Zero Degree Component) Let r be a regular value of f_0 and C a connected component of $\mathbb{F}_r$ such that $\deg(C) = 0$. Then, there exists an r-perturbation h of f such that h has no critical points in C, $h^{-1}(0) \cap C = \emptyset$. In addition, h equals f except possibly within the interior of C.

Proof: Suppose r is a regular value, then well groups $\mathsf{U}(r - \delta)$ and $\mathsf{U}(r + \delta)$ are isomorphic for all sufficiently small $\delta > 0$. Suppose $\deg(C) = \deg(f|_C) = \deg(f|_{\partial C}) = 0$. Then following the Hopf Extension Theorem ([GP74], page 145), if the function $f|_{\partial C} : \partial C \to \mathbb{S}^1$ has degree zero, then f can be extended to a globally defined map $g : C \to \mathbb{S}^1$ such that g equals to f when both are restricted to ∂C. Now we define a perturbation $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h = 0.5 \cdot f + 0.5 \cdot g$. By definition $\mathrm{d}(h, f) \leq r$, so h is an r-perturbation of f. In addition, $h^{-1}(0) \cap C$ is empty. □

Corollary 1.2 (Non-zero Degree Component) Let r be a regular value of f_0 and C a connected component of $\mathbb{F}_r$ such that $\deg(C) \neq 0$. Then for any δ-perturbation h of f, where $\delta < r$, the sum of the degrees of the critical points in $h^{-1}(0) \cap C$ is $\deg(C)$.

Proof: Suppose r is a regular value and $\deg(C) \neq 0$. We have the following commutative diagram for any δ-perturbation h of f where $\delta < r$,

$$\begin{array}{ccc} \mathsf{H}(C, \partial C) & \xrightarrow{i_*} & \mathsf{H}(C, C - h^{-1}(0)) \\ \downarrow f_*|_C & & \downarrow h_*|_0 \\ \mathsf{H}(B_r, \partial B_r) & \xrightarrow{j_*} & \mathsf{H}(B_r, B_r - \{0\}) \end{array}$$

The horizontal maps are included by space inclusions, j_* is an isomorphism, while the vertical maps are induced by f and h with restrictions, respectively. By commutativity, the sum of degrees of the critical points in $h^{-1}(0) \cap C$ is therefore $\deg(C)$. □

To prove Lemma 3.1, we note that if a critical point has static robustness r, then $r+\delta$ for an arbitrarily small δ is a regular value and $C \subseteq \mathbb{F}_{r+\delta}$ has zero degree. The lemma then follows by applying Corollary 1.1. We omit the proof of Lemma 3.2, but it follows trivially from Corollary1.2.

2. Proofs Involving Dynamic Robustness

We now give the proof for Lemma 4.1.

Proof: Suppose h is any δ-perturbation of f, that is, $\mathrm{d}(h, f) \leq \delta$. Let X_f, X_h be the sets of critical points of f and h respectively. Let bijections $\rho_f : X_f \to \mathrm{Dgm}(f_0)$ and $\rho_h : X_h \to \mathrm{Dgm}(h_0)$ be their respective dynamic robustness assignments. Recall the Stability Theorem of Well Diagrams [EMP11], we have $W_\infty(\mathrm{Dgm}(f_0), \mathrm{Dgm}(h_0)) \leq \mathrm{d}(f,h) \leq \delta$. By definition, there exists a bijection $\mu : \mathrm{Dgm}(f_0) \to \mathrm{Dgm}(g_0)$ such that for any point $a \in \mathrm{Dgm}(f_0)$, $|\mu(a) - a| \leq \delta$. Suppose $x \in X_f$ has its robustness $\rho_f(x) = r$. This implies that there exists a critical point $y \in X_h$, where $y = \rho_h^{-1} \circ \mu \circ \rho_f(x)$, such that its dynamic robustness $\rho_h(y)$ is at most δ away from r, $|\rho_f(x) - \rho_h(y)| \leq \delta$. □

Based on Lemma 4.1 we could have the following lemma that describes the correspondences between sets of critical points with respect to bounded perturbations. Its proof can be trivially derived from Lemma 4.1.

Lemma 2.1 (Dynamic Robustness Stability: A Stronger Version) Let D be the set of critical points of f with dynamic robustness greater than δ. Let h be any δ-perturbation of f, and call E its set of critical points. Then there is an injective map $i : D \to E$ such that if the dynamic robustness of x is r, then the dynamic robustness r' of $i(x)$ is $r - \delta \leq r' \leq r + \delta$.

This is a stronger statement than Lemma 4.1. With injectivity, each critical point of f has at least one critical point with similar robustness in a δ-perturbation of f as a set, rather than for each point individually. This is what allows us to pair critical points across time slices with bounded perturbations.

3. Global View of Datasets

For completeness, the global view of CentralAtlantic and SouthAtlantic datasets that include both static and dynamic robustness assignments are shown in Figure 1 and Figure 2.

References

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