Mathematic definitions and Formulas

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1 Derivatives

Definition:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \quad x \in D$$

where $f'(x_0)$ is the slope of a line tangent to function f, at x_0.
A function is said to have derivable at a point x_0 if it has finite derivate at that point, which happens if the side derivates are equal. In that case

$$f'(x_0^-) = f'(x_0^+) = f'(x_0).$$

Properties of the derivatives:

- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
- $(f \circ g)(x) = f(g(x)) = f'(g(x))g'(x)$

Derivates of some trigonometric functions:

- $(\sin u)' = u' \cos u$
- $(\cos u)' = -u' \sin u$
- $(\tan u)' = \frac{u'}{\cos^2 u}$

2 Trigonometrics

The fundamental equation of trigonometrics:

$$\sin^2(x) + \cos^2(x) = 1$$

Sine and cosine as linear combinations of complex exponentials:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad , \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Useful formulas:

$$\begin{aligned} \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \sin(a-b) &= \sin(a)\cos(b) - \cos(a)\sin(b) \\ \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \cos(a-b) &= \cos(a)\cos(b) + \sin(a)\sin(b) \end{aligned}$$

$$\begin{aligned} \sin(a) - \sin(b) &= 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right) \\ \sin(a) + \sin(b) &= 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \\ \cos(a) + \cos(b) &= 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \\ \cos(a) - \cos(b) &= -2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right) \end{aligned}$$

$$\begin{aligned} \sin(2x) &= 2\sin(x)\cos(x) \\ \cos(2x) &= 2\cos^2(x) - 1 \end{aligned}$$

3 Continuity

A function $f(x)$ is said to be continous at some point a *iff*:

$$\exists \lim_{x \to a} f(x) = f(a)$$

4 Probabilities

Some properties:

$$p(\emptyset) = 0$$

$$p(\overline{A}) = 1 - p(A)$$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

If two events are incompatibles or dijoint:

$$p(A \cap B) = 0$$

If two events are independent:

$$p(A \cap B) = p(A)p(B)$$

Conditional probability:

$$p(B/A) = \frac{p(A \cap B)}{p(A)}$$

Binomial probability law:

$$P = C_k^n p^k (1-p)^{n-k}$$

5 Counting formulas (for probabilities)

Factorial:

$$n! = n(n-1)(n-2)\cdots 1$$

Ordered combinations without repetitions:

$$^nA_p = \frac{n!}{(n-p)!}$$

Ordered combinations with repetitions:

$$^nA'_p = n^p$$

Unordered combinations without repetitions:

$$^nC_p = \frac{n!}{p!(n-p)!}$$

Some properties of unordered combinations without repetitions:

- $^nC_0 = 1$
- $^nC_n = 1$
- $^nC_1 = n$
- $^nC_p = {}^nC_{n-p}$
- $^nC_p + {}^nC_{p+1} = {}^{n+1}C_{p+1}$

6 Newton expansion

General formula:

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-1} b^2 + \cdots + {}^nC_{n-1} a b^{n-1} + {}^nC_n b^n$$

Some useful formulas from the particular case, $n = 2$:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

7 Limits

Definition:

$$\exists \lim_{x\to a} f(x) = b \quad \Leftrightarrow \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = b$$

Some trigonometric limits:

- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$
- $\lim\limits_{x\to 0} \dfrac{x}{\sin x} = 1$
- $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$
- $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1$
- $\lim\limits_{x\to 0} \dfrac{x}{\tan x} = 1$

The **Neper** number related limits:

- $\lim \left(1 + \frac{1}{n}\right)^n = e$
- $\lim \left(1 + \frac{k}{n}\right)^n = e^k$
- $\lim \left(1 - \frac{k}{n}\right)^n = e^{-k}$
- $\lim \left(1 + \frac{x}{a_n}\right)^{a_n} = e^x$, if $a_n \to \pm\infty$ and $x \in \mathbb{R}$

and the following theorems

- Theorem 1:
  If $\lim a_n = 1$ *and* $\lim b_n = +\infty$ *then,* $\lim a_n^{b_n} = e^{\lim b_n (a_n - 1)}$.
- Theorem 2:
  If $g(n) > 0$ *for all* $n \in \mathbb{N}$ *then,* $\lim g(n)^{f(n)} = \lim g(n)^{\lim f(n)}$.

8 Arithmetic progressions

Definition:

$$u_n = u_1 + (n-1) \times r$$

Sum of n elements of the progression:

$$S_n = \frac{u_1 + u_n}{2} \times n$$

9 Geometric progressions

Definition:

$$u_n = u_1 \times r^{n-1}$$

Sum of n elements of the progression:

$$S_n = u_1 \times \frac{1 - r^n}{1 - r}$$

Sum of all the elements:

$$S = \lim_{n \to \infty} S_n = \frac{u_1}{1 - r}, \text{ if } |r| < 1$$

10 Statistics for experimental data analysis

Variance of some data around the mean:

$$\sigma^2 = \frac{n \sum x_i^2 - (\sum x_i)^2}{(n-1)n^2}$$

Principle of the error propagation:

$$\Delta z = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 \Delta x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \Delta y^2 + \cdots}$$

Linear regression ($y = mx + b$):

$$m = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

$$b = \frac{(\sum x_i^2)(\sum y_i) - (\sum x_i)(\sum x_i y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

$$\Delta m = \sqrt{\frac{n \sum y_i^2 - (\sum y_i)^2 - \frac{(n \sum x_i y_i - (\sum x_i)(\sum y_i))^2}{n \sum x_i^2 - (\sum x_i)^2}}{(n-2)\left(n \sum x_i^2 - (\sum x_i)^2\right)}}$$

$$\Delta b = \sqrt{\frac{\sum x_i^2}{n}}\, \Delta m$$

$$r = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{\sqrt{\left(n \sum x_i^2 - (\sum x_i)^2\right)\left(n \sum y_i^2 - (\sum y_i)^2\right)}}$$